CBSE Class–12 Mathematics

NCERT solution

Chapter - 9

Differential Equations - Exercise 9.5

In each of the following Questions 1 to 5, show that the differential equation is homogenous and solve each of them:

1. $\left(x^2+xy\right)\ dy=\left(x^2+y^2\right)\ dx$

Ans. Given: Differential equation $\left(x^2+xy\right)\ dy=\left(x^2+y^2\right)\ dx$ ....(i)

Here degree of each coefficients of dx and dy is same therefore, it is homogenous.

$$\Rightarrow \frac{dy}{dx}=\frac{x^2+y^2}{x^2+xy}$$

$$\Rightarrow \frac{dy}{dx}=\frac{x^2\left(1+\frac{y^2}{x^2}\right)}{x^2\left(1+\frac{y}{x}\right)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1+\left(\frac{y}{x}\right)^2}{1+\left(\frac{y}{x}\right)} \quad .....(ii)$$

$$\Rightarrow \mathrm{F}\left(\frac{y}{x}\right),$$

therefore the given differential equation is homogeneous.

Putting $\frac{y}{x}=v$

$\Rightarrow \quad y = vx$

$\Rightarrow \dfrac{dy}{dx} = v.1 + x\dfrac{dv}{dx} = v + x\dfrac{dv}{dx}$

Putting value of $\dfrac{y}{x}$ and $\dfrac{dy}{dx}$ in eq. (ii), $v + x\dfrac{dv}{dx} = \dfrac{1+v^2}{1+v}$

$\Rightarrow x\dfrac{dv}{dx} = \dfrac{1+v^2}{1+v} - v$

$\Rightarrow x\dfrac{dv}{dx} = \dfrac{1+v^2-v-v^2}{1+v} = \dfrac{1-v}{1+v}$

$\Rightarrow x(1+v)\ dv = (1-v)\ dx$

$\Rightarrow \dfrac{1+v}{1-v}\ dv = \dfrac{dx}{x}$ [Separating variables]

Integrating both sides,

=> $\int \dfrac{1+v}{1-v}\ dv = \int \dfrac{1}{x}\ dx$

$\Rightarrow \int \dfrac{1+1-1+v}{1-v}\ dv = \int \dfrac{1}{x}\ dx$

$\Rightarrow \int \dfrac{2-(1-v)}{1-v}\ dv = \log x + c$

$\Rightarrow \int \left(\dfrac{2}{1-v} - 1\right)\ dv = \log x + c$

$\Rightarrow \dfrac{2\log(1-v)}{-1} - v = \log x + c$

$\Rightarrow -2\log(1-v) - v = \log x + c$

Putting $v=\frac{y}{x}$ $-2\log\left(1-\frac{y}{x}\right)-\frac{y}{x}=\log x+c$

$\Rightarrow 2\log\left(1-\frac{y}{x}\right)+\frac{y}{x}=-\log x-c$

$\Rightarrow \log\left(\frac{x-y}{x}\right)^2+\log x=-\frac{y}{x}-c$

$\Rightarrow \log\left(\frac{x-y}{x}\right)^2 .x=-\frac{y}{x}-c$

$\Rightarrow \frac{(x-y)^2}{x}=e^{\frac{-y}{x}-c}$

$\Rightarrow \frac{(x-y)^2}{x}=e^{\frac{-y}{x}}.e^{-c}$

$\Rightarrow (x-y)^2=\mathrm{C}x.e^{\frac{-y}{x}}$ where C = e^{-c}

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2. $y'=\frac{x+y}{x}$

Ans. Given: Differential equation $y'=\frac{x+y}{x}$

$\Rightarrow \frac{dy}{dx}=\frac{x}{x}+\frac{y}{x}$

$\Rightarrow \frac{dy}{dx}=1+\frac{y}{x}=f\left(\frac{y}{x}\right)$ ....(i)

Therefore, eq. (i) is homogeneous.

Putting $\frac{y}{x}=v$

$\Rightarrow \quad y = vx$

$\Rightarrow \quad \frac{dy}{dx} = v.1 + x\frac{dv}{dx} = v + x\frac{dv}{dx}$

Putting value of y and $\frac{dy}{dx}$ in eq. (i)

=> $v + x\frac{dv}{dx} = 1 + v$

$\Rightarrow \quad x\frac{dv}{dx} = 1$

$\Rightarrow \quad x\, dv = dx$

$\Rightarrow \quad dv = \frac{dx}{x}$ [Separating variables]

Integrating both sides,

=> $\int 1\, dv = \int \frac{dx}{x}$

$\Rightarrow \quad v = \log|x| + c$

Putting $\frac{y}{x} = v$,

=> $\frac{y}{x} = \log|x| + c$

$\Rightarrow \quad y = x\log|x| + xc$

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3. $(x - y)\ dy + (x + y)\ dx = 0$

Ans. Given: Differential equation $(x - y)\ dy - (x + y)\ dx = 0$ ....(i)

This given equation is homogeneous because each coefficients of dx and dy is of degree 1.

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x\left(1+\frac{y}{x}\right)}{x\left(1-\frac{y}{x}\right)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1+\frac{y}{x}}{1-\frac{y}{x}} = f\left(\frac{y}{x}\right)$$

Putting $\frac{y}{x} = v$

$$\Rightarrow y = vx$$

$$\Rightarrow \frac{dy}{dx} = v.1 + x\frac{dv}{dx} = v + x\frac{dv}{dx} \quad \text{....(ii)}$$

Putting value of y and $\frac{dy}{dx}$ in eq. (ii)

$$v + x\frac{dv}{dx} = \frac{1+v}{1-v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v}{1-v} - v$$

$$=> x\frac{dv}{dx} = \frac{1+v-v+v^2}{1-v}$$

$$\Rightarrow => x\frac{dv}{dx} = \frac{1+v^2}{1-v}$$

$$\Rightarrow x(1-v)\ dv = \left(1+v^2\right)\ dx$$

$\Rightarrow \dfrac{(1-v)}{1+v^2}\ dv = \dfrac{dx}{x}$ [Separating variables]

Integrating both sides,

=> $\displaystyle\int \frac{(1-v)}{1+v^2}\ dv = \int \frac{1}{x}\ dx$

$$\Rightarrow \int \frac{1}{1+v^2}\ dv - \int \frac{v}{1+v^2}\ dv = \int \frac{1}{x}\ dx + c$$

$$\Rightarrow \tan^{-1} v - \frac{1}{2}\int \frac{2v}{1+v^2}\ dv = \int \frac{1}{x}\ dx + c$$

$$\Rightarrow \tan^{-1} v - \frac{1}{2}\log\left(1+v^2\right) = \log x + c$$

Putting $v = \dfrac{y}{x}$,

=> $\tan^{-1}\dfrac{y}{x} - \dfrac{1}{2}\log\left(1+\dfrac{y^2}{x^2}\right) = \log x + c$

$$\Rightarrow \tan^{-1}\frac{y}{x} - \frac{1}{2}\log\left(\frac{x^2+y^2}{x^2}\right) = \log x + c$$

$$\Rightarrow \tan^{-1}\frac{y}{x} - \left[\frac{1}{2}\log\left(x^2+y^2\right) - \log x^2\right] = \log x + c$$

$$\Rightarrow \tan^{-1}\frac{y}{x} - \frac{1}{2}\log\left(x^2+y^2\right) + \frac{1}{2}\log x^2 = \log x + c$$

$$\Rightarrow \tan^{-1}\frac{y}{x} - \frac{1}{2}\log\left(x^2+y^2\right) + \frac{1}{2}.2\log x = \log x + c$$

$$\Rightarrow \tan^{-1}\frac{y}{x} - \frac{1}{2}\log\left(x^2+y^2\right) = c$$

$\Rightarrow \tan^{-1}\frac{y}{x}=\frac{1}{2}\log\left(x^2+y^2\right)+c$

4. $\left(x^2-y^2\right)\ dx+2xy\ dy=0$

Ans. Given: Differential equation $\left(x^2-y^2\right)\ dx+2xy\ dy=0$ -

This equation is homogeneous because degree of each coefficient of dx and dy is same i.e., 2

$\Rightarrow\ 2xy\ dy=-\left(x^2-y^2\right)\ dx$

$\Rightarrow \frac{dy}{dx}=\frac{-\left(x^2-y^2\right)}{2xy}=\frac{y^2-x^2}{2xy}$

$\Rightarrow \frac{dy}{dx}=\frac{\left(\frac{y}{x}\right)^2-1}{2\frac{y}{x}}=f\left(\frac{y}{x}\right)$ ....(ii)

Therefore, the given equation is homogeneous.

Put $\frac{y}{x}=v$

$\Rightarrow\ y=vx$

$\Rightarrow \frac{dy}{dx}=v.1+x\frac{dv}{dx}=v+x\frac{dv}{dx}$

Putting these values of $\frac{y}{x}$ and $\frac{dy}{dx}$ in eq. (ii), we get

$v+x\frac{dv}{dx}=\frac{v^2-1}{2v}$

$$\Rightarrow x\frac{dv}{dx}=\frac{v^2-1}{2v}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v^2-1-2v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{-v^2-1}{2v}=\frac{-\left(v^2+1\right)}{2v}$$

$$\Rightarrow x\ 2v\ dv=-\left(v^2+1\right)\ dx$$

$$\Rightarrow \frac{2v\ dv}{v^2+1}=\frac{-dx}{x}$$

Integrating both sides,

=> $\int\frac{2v}{v^2+1}\ dv=-\int\frac{1}{x}\ dx$

$$\Rightarrow \log\left(v^2+1\right)+\log x=\log c$$

$$\Rightarrow \log\left(v^2+1\right)x=\log c$$

$$\Rightarrow \left(v^2+1\right)x=c$$

Put $\frac{y}{x}=v$,

=> $\left(\frac{y^2}{x^2}+1\right)x=c$

$$\Rightarrow \left(\frac{y^2+x^2}{x^2}\right)x=c$$

$$\Rightarrow \frac{y^2+x^2}{x}=c$$

$\Rightarrow\ x^2 + y^2 = cx$

5. $x^2\left(\frac{dy}{dx}\right) = x^2 - 2y^2 + xy$

Ans. Given: Differential equation $x^2\frac{dy}{dx} = x^2 - 2y^2 + xy$

$\Rightarrow\ \frac{dy}{dx} = \frac{x^2}{x^2} - \frac{2y^2}{x^2} + \frac{xy}{x^2}$

$\Rightarrow\ \frac{dy}{dx} = 1 - 2\left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right) = f\left(\frac{y}{x}\right)$ ....(i)

Therefore, the given differential equation is homogeneous as all terms of x and y are of same degree i.e., degree 2.

Putting $\frac{y}{x} = v$

$\Rightarrow\ y = vx$

$\Rightarrow\ \frac{dy}{dx} = v.1 + x\frac{dv}{dx} = v + x\frac{dv}{dx}$

Putting these values of $\frac{y}{x}$ and $\frac{dy}{dx}$ in eq. (i), we get

$v + x\frac{dv}{dx} = 1 - 2v^2 + v$

$\Rightarrow\ x\frac{dv}{dx} = 1 - 2v^2$

$\Rightarrow\ x\ dv = \left(1 - 2v^2\right)\ dx$

$\Rightarrow \dfrac{dv}{1-2v^2} = \dfrac{dx}{x}$ [Separating variables]

Integrating both sides,

=> $\int \dfrac{1}{1-2v^2}\ dv = \int \dfrac{1}{x}\ dx$

$\Rightarrow \int \dfrac{1}{(1)^2 - \left(\sqrt{2}v\right)^2}\ dv = \int \dfrac{1}{x}\ dx$

$\Rightarrow \dfrac{1}{2.1} \dfrac{\log\left|\dfrac{1+\sqrt{2}v}{1-\sqrt{2}v}\right|}{\sqrt{2} \rightarrow \text{Cofficient of } v}$ = $\log|x| + c$

$\left[\because \int \dfrac{1}{a^2 - x^2}\ dx = \dfrac{1}{2a}\log\left|\dfrac{a+x}{a-x}\right|\right]$

Putting $\dfrac{y}{x} = v$,

=> $\dfrac{1}{2\sqrt{2}}\log\left|\dfrac{1+\sqrt{2}\dfrac{y}{x}}{1-\sqrt{2}\dfrac{y}{x}}\right|$ = $\log|x| + c$

=> $\dfrac{1}{2\sqrt{2}}\log\left|\dfrac{x+\sqrt{2}y}{x-\sqrt{2}y}\right|$ = $\log|x| + c$

In each of the Questions 6 to 10, show that the given differential equation is homogeneous and solve each of them:

6. $x\ dy - y\ dx = \sqrt{x^2 + y^2}\ dx$

Ans. Given: Differential equation $x\ dy - y\ dx = \sqrt{x^2 + y^2}\ dx$

$\Rightarrow \ x\,dy = y\,dx + \sqrt{x^2+y^2}\ dx$

$\Rightarrow x\frac{dy}{dx} = y + \sqrt{x^2+y^2}$

$\Rightarrow x\frac{dy}{dx} = y + x\sqrt{1+\left(\frac{y}{x}\right)^2}$

[Dividing by x]

$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2} = f\left(\frac{y}{x}\right)$

Therefore given differential equation is homogeneous.

Putting $\frac{y}{x} = v$

$\Rightarrow y = vx$

$\Rightarrow \frac{dy}{dx} = v.1 + x\frac{dv}{dx} = v + x\frac{dv}{dx}$

Putting these values of $\frac{y}{x}$ and $\frac{dy}{dx}$ in eq. (i), we get

=> $v + x\frac{dv}{dx} = v + \sqrt{1+v^2}$

$\Rightarrow x\frac{dv}{dx} = \sqrt{1+v^2}$

$\Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$

Integrating both sides,

$$\Rightarrow \int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$$

$$\Rightarrow \log\left(v+\sqrt{1+v^2}\right) = \log x + \log c$$

Putting $\frac{y}{x} = v$,

$$\Rightarrow \log\left(\frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2}\right) = \log xc$$

$$\Rightarrow \log\left(\frac{y+\sqrt{x^2+y^2}}{x}\right) = \log cx$$

$$\Rightarrow y+\sqrt{x^2+y^2} = cx^2$$

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7. $\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\} y\ dx = \left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\} x\ dy$

Ans. Given: Differential equation $\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\} y\ dx = \left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\} x\ dy$

$$\Rightarrow \frac{dy}{dx} = \frac{\left(x\cos\frac{y}{x} + y\sin\frac{y}{x}\right)y}{\left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)x} = \frac{xy\cos\frac{y}{x} + y^2\sin\frac{y}{x}}{xy\sin\frac{y}{x} - x^2\cos\frac{y}{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{y}{x}\cos\frac{y}{x} + \left(\frac{y}{x}\right)^2\sin\frac{y}{x}}{\frac{y}{x}\sin\frac{y}{x} - \cos\frac{y}{x}} = f\left(\frac{y}{x}\right) \quad \text{..........(i)}$$

Therefore, the given differential equation is homogeneous.

Putting $\frac{y}{x} = v$

$\Rightarrow y = vx$

$\Rightarrow \frac{dy}{dx} = v.1 + x\frac{dv}{dx} = v + x\frac{dv}{dx}$

Putting these values of $\frac{y}{x}$ and $\frac{dy}{dx}$ in eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{v\cos v + v^2 \sin v}{v\sin v - \cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + v^2 \sin v}{v\sin v - \cos v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + v^2 \sin v - v^2 \sin v + v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow x\left(v\sin v - \cos v\right)\, dv = 2v\cos v\, dx$$

$\Rightarrow \frac{v\sin v - \cos v}{v\cos v}\, dv = 2\frac{dx}{x}$ [Separating variables]

Integrating both sides,

$$\Rightarrow \int \frac{v\sin v - \cos v}{v\cos v}\, dv = 2\int \frac{1}{x}\, dx$$

$$\Rightarrow \int\left(\frac{v\sin v}{v\cos v} - \frac{\cos v}{v\cos v}\right) dv = 2\int \frac{1}{x}\, dx$$

$\Rightarrow \int\left(\tan v - \frac{1}{v}\right) dv = 2\int \frac{1}{x}\, dx$

$\Rightarrow \log|\sec v| - \log|v| = 2\log|x| + \log|c|$

$\Rightarrow \log\left|\frac{\sec v}{v}\right| = \log|x|^2 + \log|c|$

$\Rightarrow log\left|\frac{secv}{v}\right| = log|c|x^2$

$\Rightarrow \frac{\sec v}{v} = \pm|c|x^2$

$\Rightarrow \sec v = \pm|c|x^2 v$

Putting $v = \frac{y}{x}$,

=> $\sec\frac{y}{x} = Cx^2\frac{y}{x}$ where C = $\pm c$

$\Rightarrow \sec\frac{y}{x} = Cxy$

$\Rightarrow \frac{1}{\cos\frac{y}{x}} = Cxy$

$\Rightarrow Cxy.\cos\frac{y}{x} = 1$

$\Rightarrow xy.\cos\frac{y}{x} = \frac{1}{C}$

8. $x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$

Ans. Given: Differential equation $x\frac{dy}{dx}-y+x\sin\left(\frac{y}{x}\right)=0$

$\Rightarrow\ x\frac{dy}{dx}=y-x\sin\left(\frac{y}{x}\right)$

$\Rightarrow\ \frac{dy}{dx}=\frac{y}{x}-\sin\left(\frac{y}{x}\right)=f\left(\frac{y}{x}\right)$ ....(i)

Therefore, the given differential equation is homogeneous.

Putting $\frac{y}{x}=v$

$\Rightarrow\ y=vx$

$\Rightarrow\ \frac{dy}{dx}=v.1+x\frac{dv}{dx}=v+x\frac{dv}{dx}$

Putting these values of $\frac{y}{x}$ and $\frac{dy}{dx}$ in eq. (i), we get

$v+x\frac{dv}{dx}=v-\sin v$

$\Rightarrow\ x\frac{dv}{dx}=-\sin v$

$\Rightarrow\ x\,dv=-\sin v\,dx$

$\Rightarrow\ \frac{dv}{\sin v}=\frac{-dx}{x}$

$\Rightarrow\ \operatorname{cosec} v\,dv=\frac{-dx}{x}$

Integrating both sides,

=> $\int \operatorname{cosec} v \, dv = -\int \frac{1}{x} \, dx$

$\Rightarrow \log \left|\operatorname{cosec} v - \cot v\right| = -\log |x| + \log |c|$

$\Rightarrow \log \left|\operatorname{cosec} v - \cot v\right| = \log \left|\frac{c}{x}\right|$

$\Rightarrow \operatorname{cosec} v - \cot v = \pm \frac{c}{x}$

$\Rightarrow \operatorname{cosec} \frac{y}{x} - \cot \frac{y}{x} = \pm \frac{c}{x}$ [putting $v = \frac{y}{x}$]

$\Rightarrow \frac{1}{\sin \frac{y}{x}} - \frac{\cos \frac{y}{x}}{\sin \frac{y}{x}} = \frac{C}{x}$

$\Rightarrow \frac{1 - \cos \frac{y}{x}}{\sin \frac{y}{x}} = \frac{C}{x}$ where $\pm c = C$

$\Rightarrow x\left(1 - \cos \frac{y}{x}\right) = C \sin \frac{y}{x}$

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9. $y \, dx + x \log\left(\frac{y}{x}\right) dy - 2x \, dy = 0$

Ans. Given: Differential equation $y \, dx + x\left(\log \frac{y}{x}\right) dy - 2x \, dy = 0$

$\Rightarrow y \, dx = 2x \, dy - x\left(\log \frac{y}{x}\right) dy$

$$\Rightarrow\ y\ dx = x\left(2-\log\frac{y}{x}\right)\ dy$$

$$\Rightarrow\ \frac{dy}{dx} = \frac{\frac{y}{x}}{2-\log\frac{y}{x}} = f\left(\frac{y}{x}\right)$$ ....(i)

Therefore, the given differential equation is homogeneous.

Putting $\frac{y}{x} = v$

$$\Rightarrow\ y = vx$$

$$\Rightarrow\ \frac{dy}{dx} = v.1 + x\frac{dv}{dx} = v + x\frac{dv}{dx}$$

Putting these values of $\frac{y}{x}$ and $\frac{dy}{dx}$ in eq. (i), we get

$$\Rightarrow\ v + x\frac{dv}{dx} = \frac{v}{2-\log v}$$

$$\Rightarrow\ x\frac{dv}{dx} = \frac{v}{2-\log v} - v$$

$$= \frac{v-2v+v\log v}{2-\log v}$$

$$\Rightarrow\ x\frac{dv}{dx} = \frac{-v+v\log v}{2-\log v}$$

$$\Rightarrow\ x\frac{dv}{dx} = \frac{v(\log v-1)}{2-\log v}$$

$$\Rightarrow\ x(2-\log v)\ dv = v(\log v-1)\ dx$$

$$\Rightarrow \frac{2-\log v}{v(ogv-1)} = \frac{dx}{x}$$

$$\Rightarrow \frac{1-(\log v-1)}{v(\log v-1)}\ dv = \frac{dx}{x}$$

$$\Rightarrow \left[\frac{1}{v(\log v-1)} - \frac{1}{v}\right] dv = \frac{dx}{x}$$

Integrating both sides,

$$\Rightarrow \int\left[\frac{1}{v(\log v-1)} - \frac{1}{v}\right] dv = \int\frac{1}{x}\ dx$$

$$\Rightarrow \log|\log v-1| - \log|v| = \log|x| + \log|c|$$

$$\Rightarrow \log\left|\frac{\log v-1}{v}\right| = \log|cx|$$

$$\Rightarrow \left|\frac{\log v-1}{v}\right| = |cx|$$

$$\Rightarrow \frac{\log v-1}{v} = \pm cx = Cx \text{ where C = } \pm c$$

$$\Rightarrow \log v-1 = Cxv$$

$$\Rightarrow \log\frac{y}{x} - 1 = Cx\frac{y}{x} \text{ [Putting } v = \frac{y}{x}\text{]}$$

$$\Rightarrow \log\frac{y}{x} - 1 = Cy$$

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10. $\left(1+e^{x/y}\right)\ dx + e^{x/y}\left(1-\frac{x}{y}\right)\ dy = 0$

Ans. Given: Differential equation $\left(1+e^{\frac{x}{y}}\right)dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$

$\Rightarrow \left(1+e^{\frac{x}{y}}\right)\frac{dx}{dy}+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)=0$ [Dividing by dy]

$$\Rightarrow \left(1+e^{\frac{x}{y}}\right)\frac{dx}{dy}=-e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)$$

$$\Rightarrow \frac{dx}{dy}=\frac{-e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{\left(1+e^{\frac{x}{y}}\right)}=f\left(\frac{x}{y}\right) \text{ ....(i)}$$

Therefore, it is a homogeneous.

Now putting $\frac{x}{y}=v$

$$\Rightarrow x=vy$$

$$\Rightarrow \frac{dx}{dy}=v+y\frac{dv}{dy}$$

Putting these values of $\frac{x}{y}$ and $\frac{dx}{dy}$ in eq. (i), we have

$$v+y\frac{dv}{dy}=\frac{e^{v}(v-1)}{1+e^{v}}$$

$$\Rightarrow y\frac{dv}{dy}=\frac{e^{v}(v-1)}{1+e^{v}}-v$$

$\Rightarrow y\dfrac{dv}{dy} = \dfrac{ve^v - e^v - v - ve^v}{1+e^v} = \dfrac{-e^v - v}{1+e^v}$

$\Rightarrow y\left(1+e^v\right)\ dv = -\left(e^v + v\right)\ dy$

$\Rightarrow \dfrac{1+e^v}{v+e^v}\ dv = -\dfrac{dy}{y}$ [Separating variables]

Integrating both sides,

$\Rightarrow \int \dfrac{1+e^v}{v+e^v}\ dv = -\int \dfrac{1}{y}\ dy$

$\Rightarrow \log\left|v+e^v\right| = -\log\left|y\right| + \log\left|c\right|$

Now putting $v = \dfrac{x}{y}$,

$\Rightarrow \log\left|\dfrac{x}{y} + e^{\frac{x}{y}}\right| = -\log\left|y\right| + \log\left|c\right|$

$\Rightarrow \log\left|\dfrac{x}{y} + e^{\frac{x}{y}}\right| = \log\left|\dfrac{c}{y}\right|$

$\Rightarrow \left|\dfrac{x}{y} + e^{\frac{x}{y}}\right| = \left|\dfrac{c}{y}\right|$

$\Rightarrow \dfrac{x}{y} + e^{\frac{x}{y}} = \pm\dfrac{x}{y}$

$\Rightarrow x + ye^{\frac{x}{y}} =$ C where C = $\pm c$

For each of the differential equations in Questions from 11 to 15, find the particular

solution satisfying the given condition

11. $(x+y)\ dy+(x-y)\ dx=0; y=1$ **when** $x=1$

Ans. Given: Differential equation $(x+y)\ dy+(x-y)\ dx=0, y=1$ when $x=1$ .....(i)

$\Rightarrow$ (x+y)dy+(x-y)dx=0

$$\Rightarrow \frac{dy}{dx}=\frac{-(x-y)}{(x+y)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{y-x}{y+x}$$

$$\Rightarrow \frac{dy}{dx}=\frac{x\left(\frac{y}{x}-1\right)}{x\left(\frac{y}{x}+1\right)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\left(\frac{y}{x}-1\right)}{\left(\frac{y}{x}+1\right)}=f\left(\frac{y}{x}\right) \text{ ....(ii)}$$

Therefore the given differential equation is homogeneous because each coefficient of dx and dy is same i.e., degree 1.

Putting $\frac{y}{x}=v$

$$\Rightarrow y=vx$$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Putting these values of $\frac{y}{x}$ and $\frac{dv}{dx}$ in eq. (ii), we have

$$v+x\frac{dv}{dx}=\frac{v-1}{v+1}$$

$$\Rightarrow\ x\frac{dv}{dx}=\frac{v-1}{v+1}-v$$

$$\Rightarrow\ x\frac{dv}{dx}=\frac{v-1-v^2-v}{v+1}=\frac{-v^2-1}{v+1}$$

$$\Rightarrow\ x\frac{dv}{dx}=-\frac{v^2+1}{v+1}$$

$$\Rightarrow\ x(v+1)\ dv=-\left(v^2+1\right)\ dx$$

$$\Rightarrow\ \frac{v+1}{v^2+1}\ dv=-\frac{dx}{x}$$ [Separating variables]

Integrating both sides,

=> $\int\frac{v}{v^2+1}\ dv+\int\frac{1}{v^2+1}\ dv=-\int\frac{1}{x}\ dx$

$$\Rightarrow\ \frac{1}{2}\int\frac{2v}{v^2+1}\ dv+\tan^{-1}v=-\log x+c$$

$$\Rightarrow\ \frac{1}{2}\log\left(v^2+1\right)+\tan^{-1}v=-\log x+c$$

Now putting $v=\frac{y}{x}$,

$$\Rightarrow\ \frac{1}{2}\log\left(\left(\frac{y}{x}\right)^2+1\right)+\tan^{-1}\frac{y}{x}=-\log x+c$$

$$\Rightarrow\ \frac{1}{2}\log\left(\frac{y^2+x^2}{x^2}\right)+\tan^{-1}\frac{y}{x}=-\log x+c$$

$$\Rightarrow \frac{1}{2}\log\left(y^2+x^2\right)-\frac{1}{2}\log x^2+\tan^{-1}\frac{y}{x}=-\log x+c$$

$$\Rightarrow \frac{1}{2}log(y^2+x^2)-\frac{1}{2}\times 2logx+tan^{-1}\frac{y}{x}=-logx+c$$

$$\Rightarrow \frac{1}{2}\log\left(y^2+x^2\right)+\tan^{-1}\frac{y}{x}=c \text{ ....(iii)}$$

Now again given $y=1$ when $x=1$, therefore putting these values in eq. (iii),

$$\frac{1}{2}\log(1+1)+\tan^{-1}1=c$$

$$\Rightarrow c=\frac{1}{2}\log 2+\frac{\pi}{4}$$

Putting this value of c in eq. (iii), we get

$$\Rightarrow \frac{1}{2}\log\left(y^2+x^2\right)+\tan^{-1}\frac{y}{x}=\frac{1}{2}\log 2+\frac{\pi}{4}$$

$$\Rightarrow log(y^2+x^2)+2tan^{-1}\frac{y}{x}=log2+\frac{\pi}{4}$$

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12. $x^2\ dy+\left(xy+y^2\right)\ dx=0; y=1$ **when** $x=1$

Ans. Given: Differential equation $x^2\ dy+\left(xy+y^2\right)\ dx=0$

$$\Rightarrow x^2dy=-(xy+y^2)dx$$

$$\Rightarrow \frac{dy}{dx}=-\frac{y(x+y)}{x^2}=\frac{xy\left(1+\frac{y}{x}\right)}{x^2}$$

$$\Rightarrow \frac{dy}{dx}=-\frac{y}{x}\left(1+\frac{y}{x}\right)=f\left(\frac{y}{x}\right) \text{ ....(i)}$$

Therefore the given differential equation is homogeneous.

Putting $\frac{y}{x}=v$

$\Rightarrow\ y=vx$

$\Rightarrow\ \frac{dy}{dx}=v+x\frac{dv}{dx}$

Putting these values of $\frac{y}{x}$ and $\frac{dv}{dx}$ in eq. (i), we have

$\Rightarrow\ v+x\frac{dv}{dx}=-v(1+v)=-v-v^2$

$\Rightarrow\ x\frac{dy}{dx}=-v^2-2v$

$\Rightarrow\ x\frac{dy}{dx}=-v(v+2)$

$\Rightarrow\ \frac{dv}{v(v+2)}=-\frac{dx}{x}$

Integrating both sides, $\int\frac{1}{v(v+2)}\,dv=-\int\frac{1}{x}\,dx$

$\Rightarrow\ \frac{1}{2}\int\frac{2}{v(v+2)}\,dv=-\log|x|+\log|c|$

$\Rightarrow\ \frac{1}{2}\int\frac{(v+2)-v}{v(v+2)}\,dv=-\log|x|+\log|c|$

$\Rightarrow\ \int\left(\frac{1}{v}-\frac{1}{v+2}\right)dv=-2\log|x|+\log|c|$

$\Rightarrow\ \log\left|\frac{v}{v+2}\right|=\log\left|x^{-2}\right|+\log|c|$

$\Rightarrow \log\left|\frac{v}{v+2}\right| = \log\left|cx^{-2}\right|$

$\Rightarrow \frac{v}{v+2} = \pm cx^{-2}$

Putting $v = \frac{y}{x}$,

$\Rightarrow \frac{\frac{y}{x}}{\frac{y}{x}+2} = \pm cx^{-2}$

$\Rightarrow \frac{y}{y+2x} = \pm cx^{-2}$

$\Rightarrow x^2 y = C(y+2x)$ where C = $\pm c$ ....(ii)

Now putting $x=1$ and $y=1$ in eq. (ii), we get 1 = 3C $\Rightarrow$ $C = \frac{1}{3}$

Putting value of C in eq. (ii),

$x^2 y = \frac{1}{3}(y+2x)$

$\Rightarrow 3x^2 y = y+2x$

13. $\left[x\sin^2\left(\frac{y}{x}\right) - y\right] dx + x\,dy = 0; y = \frac{\pi}{4}$ **when** $x=1$

Ans. Given: Differential equation $\left(x\sin^2\frac{y}{x} - y\right) dx + x\,dy = 0; y = \frac{\pi}{4}, x=1$

$\Rightarrow x\,dy = -\left(x\sin^2\frac{y}{x} - y\right) dx$

$\Rightarrow \dfrac{dy}{dx} = -\sin^2\dfrac{y}{x} + \dfrac{y}{x} = f\left(\dfrac{y}{x}\right)$ ....(i)

Therefore, the given differential equation is homogeneous.

Putting $\dfrac{y}{x} = v$

$\Rightarrow y = vx$

$\Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Putting these values of $\dfrac{y}{x}$ and $\dfrac{dv}{dx}$ in eq. (i), we have

$\Rightarrow v + x\dfrac{dv}{dx} = -\sin^2 v + v$

$\Rightarrow x\dfrac{dv}{dx} = -\sin^2 v$

$\Rightarrow \dfrac{dv}{\sin^2 v} = -\dfrac{dx}{x}$ [Separating variables]

Integrating both sides,

$\Rightarrow \int \operatorname{cosec}^2 v\, dv = -\int \dfrac{1}{x}\, dx$

$\Rightarrow -\cot v = -\log|x| + c$

$\Rightarrow \cot v = \log|x| - c$

$\Rightarrow \cot\dfrac{y}{x} = \log|x| + c$ [Putting $\dfrac{y}{x} = v$] ....(ii)

Now putting $y = \dfrac{\pi}{4}, x = 1$ in eq. (ii), $\cot\dfrac{\pi}{4} = \log 1 - c$

$\Rightarrow c = -1$

Putting the value of c in eq. (ii),

$\cot\frac{y}{x} = \log|x| + 1$

$\Rightarrow \cot\frac{y}{x} = \log|x| + \log e$

$\Rightarrow \cot\frac{y}{x} = \log xe$

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14. $\frac{dy}{dx} - \frac{y}{x} + \cos ec\left(\frac{y}{x}\right) = 0$; $y = 0$ **when** $x = 1$

Ans. Given: Differential equation $\frac{dy}{dx} - \frac{y}{x} + \cos ec\frac{y}{x} = 0$; $y = 0$, $x = 1$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \cos ec\frac{y}{x} = f\left(\frac{y}{x}\right)$ ....(i)

Therefore, the given differential equation is homogeneous.

Putting $\frac{y}{x} = v$

$\Rightarrow y = vx$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Putting these values of $\frac{y}{x}$ and $\frac{dv}{dx}$ in eq. (i), we have

$\Rightarrow v + x\frac{dv}{dx} = v - \cos ec\ v$

$\Rightarrow x\dfrac{dv}{dx} = \dfrac{-1}{\sin v}$

$\Rightarrow \sin v\, dv = -\dfrac{dx}{x}$ [Separating variables]

Integrating both sides,

$\Rightarrow \int \sin v\, dv = -\int \dfrac{1}{x}\, dx$

$\Rightarrow -\cos v = -\log|x| + c$

$\Rightarrow \cos v = \log|x| - c$

$\Rightarrow \cos\dfrac{y}{x} = \log|x| + c$ [Putting $\dfrac{y}{x} = v$] ....(ii)

Now putting $y = 0, x = 1$ in eq. (ii), $\cos 0 = \log 1 - c$

$\Rightarrow c = -1$

Putting the value of c in eq. (ii),

$\cos\dfrac{y}{x} = \log|x| + 1$

$\Rightarrow \cos\dfrac{y}{x} = \log|x| + \log e$

$\Rightarrow \cos\dfrac{y}{x} = \log xe$

15. $2xy + y^2 - 2x^2\dfrac{dy}{dx} = 0; y = 2$ **when** $x = 1$

Ans. Given: Differential equation $2xy + y^2 - 2x^2\dfrac{dy}{dx} = 0$ ....(i)

$\Rightarrow -2x^2\dfrac{dy}{dx} = -2xy - y^2$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{-2xy}{-2x^2} - \dfrac{y^2}{-2x^2}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{1}{2}\left(\dfrac{y}{x}\right)^2 = f\left(\dfrac{y}{x}\right)$ ....(ii)

Therefore the given differential equation is homogeneous because each coefficient of dx and dy is same.

Putting $\dfrac{y}{x} = v$

$\Rightarrow y = vx$

$\Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Putting these values of $\dfrac{y}{x}$ and $\dfrac{dv}{dx}$ in eq. (ii), we have

$v + x\dfrac{dv}{dx} = v + \dfrac{1}{2}v^2$

$\Rightarrow x\dfrac{dv}{dx} = \dfrac{1}{2}v^2$

$\Rightarrow 2x\, dv = v^2\, dx$

$\Rightarrow 2\dfrac{dv}{v^2} = \dfrac{dx}{x}$ [Separating variables]

Integrating both sides,

$2\int v^{-2}\, dv = \int \dfrac{1}{x}\, dx$

$\Rightarrow 2\frac{v^{-1}}{-1} = \log|x| + c$

$\Rightarrow \frac{-2}{v} = \log|x| + c$

$\Rightarrow \frac{-2}{\left(\frac{y}{x}\right)} = \log|x| + c$ [Putting $\frac{y}{x} = v$]

Now putting $y = 2, x = 1$ in $\frac{-2}{\left(\frac{y}{x}\right)} = \log|x| + c$, $\frac{-2}{2} = \log 1 + c$

$\Rightarrow c = -1$

Again putting $c = -1$, in $\frac{2x}{y} = \log|x| + c$, we get

$\Rightarrow \frac{-2}{\left(\frac{y}{x}\right)} = \log|x| + c$

$\Rightarrow y(\log|x| - 1) = -2x$

$\Rightarrow y = \frac{-2x}{\log|x| - 1}$

$\Rightarrow y = \frac{2x}{1 - \log|x|}$

Choose the correct answer:

16. A homogeneous differential equation of the form $\frac{dy}{dx} = h\left(\frac{x}{y}\right)$ can be solved by making the substitution:

(A) $y = vx$

(B) $v = yx$

(C) $x = vy$

(D) $x = v$

Ans. We know that a homogeneous differential equation of the form $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$ can be solved by the substitution $\frac{x}{y} = v$ i.e., $x = vy$.

Therefore, option (C) is correct.

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17. Which of the following is a homogeneous differential equation:

(A) $(4x+6y+5)\ dy - 3(3y+2x+4)\ dx = 0$

(B) $(xy)\ dx - (x^3+y^3)\ dy = 0$

(C) $(x^3+2y^2)\ dx + 2xy\ dy = 0$

(D) $y^2\ dx + (x^2 - xy - y^2)\ dy = 0$

Ans. D