CBSE Class–12 Mathematics

NCERT solution

Chapter - 11

Three Dimensional Geometry - Exercise 11.3

Formula for equation number 1 and 2

If p is the length of perpendicular from the origin to a plane and $\hat{n}$ is a unit normal vector to the plane, then equation of the plane is $\vec{r}\,.\,\hat{n} = p$ (where of course p being length is > 0)

1. In each of the following cases, determine the direction cosines of the normal to the plane and the distance from the origin.

(a) $z = 2$

(b) $x + y + z = 1$

(c) $2x + 3y - z = 5$

(d) $5y + 8 = 0$

Ans. (a) Given: Equation of the plane is $z = 2$

Therefore, the direction ratios of the normal to the plane are 0, 0, 1.

$\Rightarrow\; a = 0, b = 0, c = 1$

$\Rightarrow\; \sqrt{a^2 + b^2 + c^2} = \sqrt{(0)^2 + (0)^2 + (1)^2} = 1$

Therefore, $\frac{0x}{1} + \frac{0y}{1} + \frac{z}{1} = \frac{2}{1}$

Comparing with $lx + my + nz = p$, we get $p = \frac{2}{1}$

$p = 2$

Therefore, direction cosines of normal to the plane are coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\hat{n}$, i.e., 0, 0, 1 and length of perpendicular from the origin to the plane is $p=2$.

(b) Given: Equation of the plane is $x+y+z=1$

$\Rightarrow a=1, b=1, c=1$

$\sqrt{a^2+b^2+c^2} = \sqrt{(1)^2+(1)^2+(1)^2} = \sqrt{3}$

Therefore, $\frac{x}{\sqrt{3}}+\frac{y}{\sqrt{3}}+\frac{z}{\sqrt{3}}=\frac{1}{\sqrt{3}}$

Comparing with $lx+my+nz=p$, we get $p=\frac{1}{\sqrt{3}}$

Therefore direction cosines of the normal to the plane are the coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\hat{n}$,

i.e., $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and length of perpendicular from the origin to the plane is $p=\frac{1}{\sqrt{3}}$.

(c) Equation of the plane is $2x+3y-z=5$

$\Rightarrow a=2, b=3, c=-1$

$\sqrt{a^2+b^2+c^2} = \sqrt{(2)^2+(3)^2+(-1)^2} = \sqrt{14}$

Therefore, $\frac{2x}{\sqrt{14}}+\frac{3y}{\sqrt{14}}-\frac{z}{\sqrt{14}}=\frac{5}{\sqrt{14}}$

Comparing with $lx+my+nz=p$, we get

$$p=\frac{5}{\sqrt{14}}$$

Therefore direction cosines of the normal to the plane are the coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\hat{n}$,

i.e., $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}$ and length of perpendicular from the origin to the plane is $p=\frac{5}{\sqrt{14}}$.

(d) Given: Equation of the plane is $5y+8=0 \Rightarrow 5y=-8 \Rightarrow -5y=8$

$\Rightarrow a=0, b=-5, c=0$

$\sqrt{a^2+b^2+c^2} = \sqrt{(0)^2+(-5)^2+(0)^2} = \sqrt{25} = 5$

Therefore, $\frac{0x}{5} + \frac{-5y}{5} + \frac{0z}{5} = \frac{8}{5}$

Comparing with $lx + my + cz = p$, we get

$$p = \frac{8}{5}$$

Therefore direction cosines of the normal to the plane are the coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\hat{n}$, i.e., $0, -1, 0$ and length of perpendicular from the origin to the plane is $p = \frac{8}{5}$.

2. Find the vector equation of a plane which is at a distance of 7 units from the origin and normal to the vector $3\hat{i} + 5\hat{j} - 6\hat{k}$.

Ans. Here $\vec{n} = 3\hat{i} + 5\hat{j} - 6\hat{k}$

$\therefore$ The unit vector perpendicular to the plane is

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{(3)^2 + (5)^2 + (-6)^2}} = \frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{70}}$$

Also $p = 7$ (given)

Therefore the equation of the required plane is $\vec{r}.\hat{n} = p$

$$\Rightarrow \vec{r}.\frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{70}} = 7$$

$$\Rightarrow \vec{r}.\left(3\hat{i} + 5\hat{j} - 6\hat{k}\right) = 7\sqrt{70}$$

3. Find the Cartesian equation of the following planes:

(a) $\vec{r}.\left(\hat{i}+\hat{j}-\hat{k}\right)=2$

(b) $\vec{r}.\left(2\hat{i}+3\hat{j}-4\hat{k}\right)=1$

(c) $\vec{r}.\left[(s-2t)\hat{i}+(3-t)\hat{j}+(2s+t)\hat{k}\right]=15$

Ans. (a) Vector equation of the plane is $\vec{r}.\left(\hat{i}+\hat{j}-\hat{k}\right)=2$ ....(i)

Putting $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ in eq. (i) as in 3-D, Cartesian equation of the plane is

$\left(x\hat{i}+y\hat{j}+z\hat{k}\right).\left(\hat{i}+\hat{j}-\hat{k}\right)=2$

$\Rightarrow\ x(1)+y(1)+z(-1)=2 \Rightarrow\ x+y-z=2$

(b) Since, $\vec{r}$ is the position vector of any arbitrary point P(x, y, z) on the plane.

$\therefore\ \left(x\hat{i}+y\hat{j}+z\hat{k}\right).\left(2\hat{i}+3\hat{j}-4\hat{k}\right)=1 \Rightarrow 2x+3y-4z=1$

which is the required Cartesian equation.

(c) Vector equation of the plane is $\vec{r}.\left[(s-2t)\hat{i}+(3-t)\hat{j}+(2s+t)\hat{k}\right]=15$

Since, Since, $\vec{r}$ is the position vector of any arbitrary point P(x, y, z) on the plane.

$\therefore\ \left(x\hat{i}+y\hat{j}+z\hat{k}\right).\left((s-2t)\hat{i}+(3-t)\hat{j}+(2s+t)\hat{k}\right)=15$

$\Rightarrow (s-2t)x+(3-t)y+(2s+t)z=15$ which is the required Cartesian equation.

4. In the following cases, find the coordinates of the foot of the perpendicular drawn from the origin:

(a) $2x+3y+4z-12=0$

(b) $3y+4z-6=0$

(c) $x+y+z=1$

(d) $5y+8=0$

Ans. (a) Given: Equation of the plane is $2x+3y+4z-12=0$ ....(i) and point is O (0, 0, 0)

Let M be the foot of the perpendicular drawn from the origin (0, 0, 0) to the given plane.

Since, direction ratios of perpendicular OM to plane are coefficients of x, y, z in $2x+3y+4z-12=0$, i.e., 2, 3, 4 = a, b, c (say)

$\therefore$ Equation of the perpendicular OM is $\frac{x-0}{2}=\frac{y-0}{3}=\frac{z-0}{4}=\lambda$ (say)

$\Rightarrow \frac{x}{2}=\frac{y}{3}=\frac{z}{4}=\lambda \Rightarrow \frac{x}{2}=\lambda, \frac{y}{3}=\lambda, \frac{z}{4}=\lambda$

$\Rightarrow x=2\lambda, y=3\lambda, z=4\lambda$

Therefore, point M on this line OM is M$(2\lambda, 3\lambda, 4\lambda)$ ....(ii)

But point M lies on plane (i)

$\therefore$ Putting $x=2\lambda, y=3\lambda, z=4\lambda$ in eq. (i), we have

$2(2\lambda)+3(3\lambda)+4(4\lambda)-12=0$

$\Rightarrow 4\lambda+9\lambda+16\lambda=12 \Rightarrow 29\lambda=12 \Rightarrow \lambda=\frac{12}{29}$

Hence, putting $\lambda=\frac{12}{29}$ in equation (ii), the coordinates of foot of the perpendicular is $\left(\frac{24}{29}, \frac{36}{29}, \frac{48}{29}\right)$.

(b) Given: Equation of the plane is $3y+4z-6=0$ ....(i) and point is O (0, 0, 0)

Let M be the foot of the perpendicular drawn from the origin (0, 0, 0) to the given plane.

Since, direction ratios of perpendicular OM to plane are coefficients of x, y, z in $3y+4z-6=0$, i.e., 0, 3, 4 = a, b, c (say)

$\therefore$ Equation of the perpendicular OM is $\frac{x-0}{0}=\frac{y-0}{3}=\frac{z-0}{4}=\lambda$ (say)

$$\Rightarrow \frac{x}{0}=\frac{y}{3}=\frac{z}{4}=\lambda \Rightarrow \frac{x}{0}=\lambda, \frac{y}{3}=\lambda, \frac{z}{4}=\lambda$$

$$\Rightarrow x=0, y=3\lambda, z=4\lambda$$

Therefore, point M on this line OM is M$(0, 3\lambda, 4\lambda)$ ....(ii)

But point M lies on plane (i)

$\therefore$ Putting $x=0, y=3\lambda, z=4\lambda$ in eq. (i), we have

$$3(3\lambda)+4(4\lambda)-6=0$$

$$\Rightarrow 9\lambda+16\lambda=6 \Rightarrow 25\lambda=6 \Rightarrow \lambda=\frac{6}{25}$$

Hence, putting $\lambda=\frac{6}{25}$ in equation (ii), the coordinates of foot of the perpendicular is $\left(0, \frac{18}{25}, \frac{24}{25}\right)$.

(c) Given: Equation of the plane is $x+y+z=1$ ....(i) and point is O (0, 0, 0)

Let M be the foot of the perpendicular drawn from the origin (0, 0, 0) to the given plane.

Since, direction ratios of perpendicular OM to plane are coefficients of x, y, z in $x+y+z=1$, i.e., 1, 1, 1 = a, b, c (say)

$\therefore$ Equation of the perpendicular OM is $\frac{x-0}{1}=\frac{y-0}{1}=\frac{z-0}{1}=\lambda$ (say)

$\Rightarrow \dfrac{x}{1}=\dfrac{y}{1}=\dfrac{z}{1}=\lambda \Rightarrow \dfrac{x}{1}=\lambda, \dfrac{y}{1}=\lambda, \dfrac{z}{1}=\lambda$

$\Rightarrow x=\lambda, y=\lambda, z=\lambda$

Therefore, point M on this line OM is M$(\lambda, \lambda, \lambda)$ ....(ii)

But point M lies on plane (i)

$\therefore$ Putting $x=\lambda, y=\lambda, z=\lambda$ in eq. (i), we have

$\lambda+\lambda+\lambda=1$

$\Rightarrow 3\lambda=1 \Rightarrow \lambda=\dfrac{1}{3}$

Hence, putting $\lambda=\dfrac{1}{3}$ in equation (ii), the coordinates of foot of the perpendicular is $\left(\dfrac{1}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right)$.

(d) Given: Equation of the plane is $5y+8=0$ ....(i) and point is O (0, 0, 0)

Let M be the foot of the perpendicular drawn from the origin (0, 0, 0) to the given plane.

Since, direction ratios of perpendicular OM to plane are coefficients of x, y, z in $5y+8=0$, i.e., 0, 5, 0 = a, b, c (say)

$\therefore$ Equation of the perpendicular OM is $\dfrac{x-0}{0}=\dfrac{y-0}{5}=\dfrac{z-0}{0}=\lambda$ (say)

$\Rightarrow \dfrac{x}{0}=\dfrac{y}{5}=\dfrac{z}{0}=\lambda \Rightarrow \dfrac{x}{0}=\lambda, \dfrac{y}{5}=\lambda, \dfrac{z}{0}=\lambda$

$\Rightarrow x=0, y=5\lambda, z=0$

Therefore, point M on this line OM is M$(0, 5\lambda, 0)$ ....(ii)

But point M lies on plane (i)

$\therefore$ Putting $x=0, y=5\lambda, z=0$ in eq. (i), we have

$0+5\times 5\lambda+0=-8$

$\Rightarrow 25\lambda=-8 \Rightarrow \lambda=\frac{-8}{25}$

Hence, putting $\lambda=\frac{-8}{25}$ in equation (ii), the coordinates of foot of the perpendicular is $\left(0,\frac{-40}{25},0\right)=\left(0,\frac{-8}{5},0\right)$.

5. Find the vector and Cartesian equations of the planes

(a) that passes through the point $(1,0,-2)$ and the normal to the plane is $\hat{i}+\hat{j}-\hat{k}$.

(b) that passes through the point (1, 4, 6) and the normal vector to the plane is $\hat{i}-2\hat{j}+\hat{k}$.

Ans. (a) Vector form: The given point on the plane is $(1,0,-2)$

$\therefore$ The position vector of the given point is $\vec{a}=(1,0,-2)=\hat{i}+0\hat{j}-2\hat{k}=\hat{i}-2\hat{k}$

Also Normal vector to the plane is $\vec{n}=\hat{i}+\hat{j}-\hat{k}$

$\therefore$ Vector equation of the required line is $\left(\vec{r}-\vec{a}\right).\vec{n}=0$

$\Rightarrow \vec{r}.\vec{n}-\vec{a}.\vec{n}=0 \Rightarrow \vec{r}.\vec{n}=\vec{a}.\vec{n}$

Putting the values of $\vec{a}$ and $\vec{n}$,

$\vec{r}.\left(\hat{i}+\hat{j}-\hat{k}\right)=\left(\hat{i}-2\hat{k}\right).\left(\hat{i}+\hat{j}-\hat{k}\right) \Rightarrow \vec{r}.\left(\hat{i}+\hat{j}-\hat{k}\right)=1(1)+0(1)+(-2)(-1)$

$\Rightarrow \vec{r}.\left(\hat{i}+\hat{j}-\hat{k}\right)=1+2 \Rightarrow \vec{r}.\left(\hat{i}+\hat{j}-\hat{k}\right)=3$

Cartesian form: The plane passes through the point $(1, 0, -2)$ = (x_1, y_1, z_1)

Normal vector to the plane is $\vec{n} = \hat{i} + \hat{j} - \hat{k}$

$\therefore$ Direction ratios of normal to the plane are coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\vec{n}$ are $1, 1, -1$

$\therefore$ Cartesian form of equation of plane is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

$\Rightarrow 1(x-1) + 1(y-0) - (z+2) = 0 \Rightarrow x - 1 + y - z - 2 = 0$

$\Rightarrow x + y - z = 3$

(b) Vector form: The given point on the plane is $(1, 4, 6)$

$\therefore$ The position vector of the given point is $\vec{a} = (1, 4, 6) = \hat{i} + 4\hat{j} + 6\hat{k}$

Also Normal vector to the plane is $\vec{n} = \hat{i} - 2\hat{j} + \hat{k}$

$\therefore$ Vector equation of the required line is $(\vec{r} - \vec{a}).\vec{n} = 0$

$\Rightarrow \vec{r}.\vec{n} - \vec{a}.\vec{n} = 0 \Rightarrow \vec{r}.\vec{n} = \vec{a}.\vec{n}$

Putting the values of $\vec{a}$ and $\vec{n}$,

$\vec{r}.(\hat{i} - 2\hat{j} + \hat{k}) = (\hat{i} + 4\hat{j} + 6\hat{k}).(\hat{i} - 2\hat{j} + \hat{k}) \Rightarrow \vec{r}.(\hat{i} + \hat{j} - \hat{k}) = 1(1) + 4(-2) + (6)(1)$

$\Rightarrow \vec{r}.(\hat{i} + \hat{j} - \hat{k}) = 1 - 8 + 6 \Rightarrow \vec{r}.(\hat{i} + \hat{j} - \hat{k}) = -1$

Cartesian form: The plane passes through the point $(1, 4, 6)$ = (x_1, y_1, z_1)

Normal vector to the plane is $\vec{n} = \hat{i} - 2\hat{j} + \hat{k}$

$\therefore$ Direction ratios of normal to the plane are coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\vec{n}$ are $1, -2, 1$

$\therefore$ Cartesian form of equation of plane is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

$\Rightarrow 1(x-1)-2(y-4)+1(z-6)=0 \Rightarrow x-1-2y+8+z-6=0$

$\Rightarrow x-2y+z=-1$

6. Find the equations of the planes that passes through three points:

(a) $(1,1,-1),(6,4,-5),(-4,-2,3)$

(b) $(1,1,0),(1,2,1),(-2,2,-1)$

Ans. We know that through three collinear points A, B, C i.e., through a straight line, we can pass an infinite number of planes.

(a) The three given points are A$(1,1,-1)$, B$(6,4,-5)$ and C$(-4,-2,3)$

Now direction ratios of line AB are $6-1, 4-1, -5+1$ $[\because x_2-x_1, y_2-y_1, z_2-z_1]$

= $5, 3, -4 = a_1, b_1, c_1$ (say)

Again direction ratios of line BC are $-4-6, -2-4, 3-(-5)$

= $-10, -6, 8 = a_2, b_2, c_2$ (say)

Now $\frac{a_1}{a_2}=\frac{5}{-10}, \frac{b_1}{b_2}=\frac{3}{-6}, \frac{c_1}{c_2}=\frac{-4}{8} \Rightarrow \frac{a_1}{a_2}=\frac{-1}{2}, \frac{b_1}{b_2}=\frac{-1}{2}, \frac{c_1}{c_2}=\frac{-1}{2}$

Since, $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

Therefore, line AB and BC are parallel and B is their common point.

$\therefore$ Points A, B and C are collinear and hence an infinite number of planes can be drawn through the three given collinear points i.e. no unique plane can be drawn.

(b) The three given points are A$(1,1,0)$, B$(1,2,1)$ and C$(-2,2,-1)$

Now direction ratios of line AB are $1-1, 2-1, 1-0$ $[\because x_2-x_1, y_2-y_1, z_2-z_1]$

= $0, 1, 1 = a_1, b_1, c_1$ (say)

Again direction ratios of line BC are $-2-1, 2-2, -1-1$

= $-3, 0, -2 = a_2, b_2, c_2$ (say)

Now $\frac{a_1}{a_2} = \frac{0}{-3}, \frac{b_1}{b_2} = \frac{1}{0}, \frac{c_1}{c_2} = \frac{1}{-2} \Rightarrow \frac{a_1}{a_2} = 0, \frac{b_1}{b_2} = \infty, = \frac{c_1}{c_2} = \frac{-1}{2}$

Since, $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

$\therefore$ Points A, B and C are not collinear and hence the unique plane can be drawn through the three given collinear points, i.e.,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-1 & z-0 \\ 1-1 & 2-1 & 1-0 \\ -2-1 & 2-1 & -1-0 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-1 & z \\ 0 & 1 & 1 \\ -3 & 1 & -1 \end{vmatrix} = 0$$

Expanding along first row,

$$\Rightarrow (x-1)(-1-1)-(y-1)(0+3)+z(0+3)=0$$

$$\Rightarrow -2(x-1)-3(y-1)+3z=0$$

$$\Rightarrow -2x+2-3y+3+3z=0$$

$$\Rightarrow 2x-3y+3z+5=0$$

$\Rightarrow 2x+3y-3z=5$

Hence the equation of required plane is $2x+3y-3z=5$.

7. Find the intercepts cut off by the plane $2x+y-z=5$.

Ans. Equation of the plane is $2x+y-z=5$

$$\Rightarrow \frac{2x}{5}+\frac{y}{5}-\frac{z}{5}=1$$

$$\Rightarrow \frac{x}{\left(\frac{5}{2}\right)}+\frac{y}{5}-\frac{z}{5}=1$$

Comparing with intercept form $\frac{x}{a}+\frac{y}{b}-\frac{z}{c}=1$, we have $a=\frac{5}{2}, b=5, c=-5$ which are intercepts cut off by the plane on $x-$ axis, $y-$ axis and $z-$ axis respectively.

8. Find the equation of the plane with intercept 3 on the $y-$ axis and parallel to ZOX plane.

Ans. Since equation of ZOX plane is $y=0$.

$\therefore$ Equation of any plane parallel to ZOX plane is $y=k$ ....(i)

[$\because$ Equation of any plane parallel to the plane $ax+by+cz+d=0$ is $ax+by+cz+k=0$ i.e., change only the constant term]

Now, Plane (i) makes an intercept 3 on the $y-$axis ($\Rightarrow x=0$ and $z=0$) i.e., plane (i) passes through (0, 3, 0).

Putting $x=0, y=3$ and $z=0$ in eq. (i), $3=k$

Putting $k=3$ in eq. (i), equation of required plane is $y=3$.

9. Find the equation of the plane through the intersection of the planes

$3x-y+2z-4=0$ **and** $x+y+z-2=0$ **and the point (2, 2, 1).**

Ans. Equations of given planes are $3x-y+2z-4=0$ and $x+y+z-2=0$

Since, equation of any plane through the intersection of these two planes is

L.H.S. of plane I + λ (L.H.S. of plane II) = 0

$\Rightarrow 3x-y+2z-4+\lambda(x+y+z-2)=0$ ....(i)

Now, required plane (i) passes through the point (2, 2, 1).

Putting $x=2, y=2, z=1$ in eq. (i),

$3\times2-2+2\times1-4+\lambda(2+2+1-2)=0$

$\Rightarrow 6-2+2-4+\lambda(2+2+1-2)=0$

$\Rightarrow 2+3\lambda=0$

$\Rightarrow \lambda=\frac{-2}{3}$

Now putting $\lambda=\frac{-2}{3}$ in eq. (i) of required plane is

$3x-y+2z-4+\frac{-2}{3}(x+y+z-2)=0$

$\Rightarrow 9x-3y+6z-12-2z-2y-2z+4=0$

$\Rightarrow 7x-5y+4z-8=0$

10. Find the vector equation of the plane passing through the intersection of the planes $\vec{r}.(2\hat{i}+2\hat{j}-3\hat{k})=7$, $\vec{r}.(2\hat{i}+5\hat{j}+3\hat{k})=9$ **and through the point (2, 1, 3).**

Ans. Equation of first plane is $\vec{r}.(2\hat{i}+2\hat{j}-3\hat{k})=7$

$\Rightarrow \vec{r}.\left(2\hat{i}+2\hat{j}-3\hat{k}\right)-7=0$ ....(i)

Again equation of the second plane is $\vec{r}.\left(2\hat{i}+5\hat{j}+3\hat{k}\right)=9$

$\Rightarrow \vec{r}.\left(2\hat{i}+5\hat{j}+3\hat{k}\right)-9=0$ ....(ii)

Since, equation of any plane passing through the line of intersection of two planes is

L.H.S. of plane I + λ (L.H.S. of plane II) = 0

$\Rightarrow \vec{r}.\left(2\hat{i}+2\hat{j}-3\hat{k}\right)-7+\lambda\left\{\vec{r}.\left(2\hat{i}+5\hat{j}+3\hat{k}\right)-9\right\}=0$ ....(iii)

Now, the plane (iii) passes through the point (2, 1, 3) = (x, y, z)

$\therefore \vec{r}=x\hat{i}+y\hat{j}+z\hat{k}=2\hat{i}+\hat{j}+3\hat{k}$

Putting this value of $\vec{r}$ in eq. (iii),

$\left(2\hat{i}+\hat{j}+3\hat{k}\right).\left(2\hat{i}+2\hat{j}-3\hat{k}\right)-7+\lambda\left\{\left(2\hat{i}+\hat{j}+3\hat{k}\right).\left(2\hat{i}+5\hat{j}+3\hat{k}\right)-9\right\}=0$

$\Rightarrow 4+2-9-7+\lambda(4+5+9-9)=0$

$\Rightarrow -10+9\lambda=0 \Rightarrow \lambda=\frac{10}{9}$

Putting $\lambda=\frac{10}{9}$ in eq. (iii) of required plane is

$\vec{r}.\left(2\hat{i}+2\hat{j}-3\hat{k}\right)-7+\frac{10}{9}\left\{\vec{r}.\left(2\hat{i}+5\hat{j}+3\hat{k}\right)-9\right\}=0$

$\Rightarrow \vec{r}.\left(18\hat{i}+18\hat{j}-27\hat{k}\right)-63+\left\{\vec{r}.\left(20\hat{i}+50\hat{j}+30\hat{k}\right)-90\right\}=0$

$\Rightarrow \vec{r}\left(18\hat{i}+18\hat{j}-27\hat{k}+20\hat{i}+50\hat{j}+30\hat{k}\right)-153=0$

$\Rightarrow \vec{r}.\left(38\hat{i}+68\hat{j}+3\hat{k}\right)=153$

11. Find the equation of the plane through the line of intersection of the planes $x+y+z=1$ and $2x+3y+4z=5$ which is perpendicular to the plane $x-y+z=0$.

Ans. Equations of the given planes are $x+y+z=1$ and $2x+3y+4z=5$

$\Rightarrow x+y+z-1=0$ and $2x+3y+4z-5=0$

Since, equation of any plane passing through the line of intersection of two planes is

L.H.S. of plane I + λ (L.H.S. of plane II) = 0

$\Rightarrow x+y+z-1+\lambda\left(2x+3y+4z-5\right)=0$ ....(i)

$\Rightarrow x+y+z-1+2\lambda x+3\lambda y+4\lambda z-5\lambda=0$

$\Rightarrow \left(1+2\lambda\right)x+\left(1+3\lambda\right)y+\left(1+4\lambda\right)z-1-5\lambda=0$

According to the question, this plane is perpendicular to the plane $x-y+z=0$

$\therefore\ a_1a_2+b_1b_2+c_1c_2=0$

$\Rightarrow \left(1+2\lambda\right)-\left(1+3\lambda\right)+1+4\lambda=0$

$\Rightarrow 1+2\lambda-1-3\lambda+1+4\lambda=0$

$\Rightarrow 3\lambda+1=0 \Rightarrow \lambda=\frac{-1}{3}$

Putting $\lambda=\frac{-1}{3}$ in eq. (i) of required plane is

$x+y+z-1+\frac{-1}{3}\left(2x+3y+4z-5\right)=0$

$\Rightarrow 3x+3y+3z-3-2x-3y-4z+5=0$

$\Rightarrow \ x - z + 2 = 0$

12. Find the angle between the planes whose vector equations are $\vec{r}.\left(2\hat{i} + 2\hat{j} - 3\hat{k}\right) = 5$ and $\vec{r}.\left(3\hat{i} - 3\hat{j} + 5\hat{k}\right) = 3$.

Ans. Equation of one plane is $\vec{r}.\left(2\hat{i} + 2\hat{j} - 3\hat{k}\right) = 5$ ....(i)

Comparing this equation with $\vec{r}.\vec{n_1} = d_1$, we have

Normal vector to plane (i) is $\vec{n_1} = 2\hat{i} + 2\hat{j} - 3\hat{k}$

Again, equation of second plane is $\vec{r}.\left(3\hat{i} - 3\hat{j} + 5\hat{k}\right) = 3$ ....(ii)

Comparing this equation with $\vec{r}.\vec{n_1} = d_1$, we have

Normal vector to plane (i) is $\vec{n_1} = 3\hat{i} - 3\hat{j} + 5\hat{k}$

Let θ be the acute angle between plane (i) and (ii).

$\therefore$ angle between normals $\vec{n_1}$ and $\vec{n_2}$ to planes (i) and (ii) is also θ.

$$\therefore \ \cos\theta = \frac{\left|\vec{n_1}\vec{n_2}\right|}{\left|\vec{n_1}\right|.\left|\vec{n_2}\right|} = \frac{\left|2(3) + 2(-3) + (-3)5\right|}{\sqrt{4+4+9}\sqrt{9+9+25}}$$

$$= \frac{\left|6 - 6 - 15\right|}{\sqrt{17}\sqrt{43}}$$

$$= \frac{\left|-15\right|}{\sqrt{731}} = \frac{15}{\sqrt{731}}$$

$$\Rightarrow \ \theta = \cos^{-1}\frac{15}{\sqrt{731}}$$

13. In the following cases, determine whether the given planes are parallel or perpendicular and in case they are neither, find the angle between them.

(a) $7x+5y+6z+30=0$ **and** $3x-y-10z+4=0$

(b) $2x+y+3z-2=0$ **and** $x-2y+5=0$

(c) $2x-2y+4z+5=0$ **and** $3x-3y+6z-1=0$

(d) $2x-y+3z-1=0$ **and** $2x-y+3z+3=0$

(e) $4x+8y+z-8=0$ **and** $y+z-4=0$

Ans. (a) Equations of the given planes are $7x+5y+6z+30=0$ $\left(a_1x+b_1y+c_1z+d_1=0\right)$ and

$3x-y-10z+4=0$ $\left(a_2x+b_2y+c_2z+d_2=0\right)$

Here, $\frac{a_1}{a_2}=\frac{7}{3}, \frac{b_1}{b_2}=\frac{5}{-1}, \frac{c_1}{c_2}=\frac{6}{-10}$

Since $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$

Therefore, the given two planes are not parallel.

Again $a_1a_2+b_1b_2+c_1c_2$ = 21 – 5 – 60 = 21 – 65 = –44

Since $a_1a_2+b_1b_2+c_1c_2\neq 0$

Therefore, the given two planes are not perpendicular.

Now let θ be the angle between the two planes.

$$\therefore \quad \cos\theta=\frac{\left|a_1a_2+b_1b_2+c_1c_2\right|}{\sqrt{a_1^2+b_1^2+c_1^2}.\sqrt{a_2^2+b_2^2+c_2^2}}$$

= $\frac{|7(3)+5(-1)+6(-10)|}{\sqrt{(7)^2+(5)^2+(6)^2}.\sqrt{(3)^2+(-1)^2+(-10)^2}}$

= $\frac{|21-5-60|}{\sqrt{49+25+36}.\sqrt{9+1+100}}$

= $\frac{|-44|}{\sqrt{110}.\sqrt{110}}$ = $\frac{44}{110}$ = $\frac{2}{5}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{2}{5}\right)$

(b) equations of the given planes are $2x+y+3z-2=0$ $(a_1x+b_1y+c_1z+d_1=0)$ and $x-2y+5=0$ i.e., $x-2y+0z+5=0$ $(a_2x+b_2y+c_2z+d_2=0)$

Here, $\frac{a_1}{a_2}=\frac{2}{1}, \frac{b_1}{b_2}=\frac{1}{-2}, \frac{c_1}{c_2}=\frac{3}{0}$

Since $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

Therefore, the given two planes are not parallel.

Again $a_1a_2+b_1b_2+c_1c_2$ = $2(1)+1(-2)+3(0)$ = 2 – 2 + 0 = 0

Since $a_1a_2+b_1b_2+c_1c_2=0$

Therefore, the given two planes are perpendicular.

(c) equations of the given planes are $2x-2y+4z+5=0$ $(a_1x+b_1y+c_1z+d_1=0)$ and $3x-3y+6z-1=0$ $(a_2x+b_2y+c_2z+d_2=0)$

Here, $\frac{a_1}{a_2}=\frac{2}{3}, \frac{b_1}{b_2}=\frac{-2}{-3}=\frac{2}{3}, \frac{c_1}{c_2}=\frac{4}{6}=\frac{2}{3}$

Since $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Therefore, the given two planes are parallel.

(d) equations of the given planes are $2x - y + 3z - 1 = 0$ $(a_1x + b_1y + c_1z + d_1 = 0)$ and $2x - y + 3z + 3 = 0$ $(a_2x + b_2y + c_2z + d_2 = 0)$

Here, $\frac{a_1}{a_2} = \frac{2}{2} = \frac{1}{1}, \frac{b_1}{b_2} = \frac{-1}{-1} = \frac{1}{1}, \frac{c_1}{c_2} = \frac{3}{3} = \frac{1}{1}$

Since $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Therefore, the given two planes are parallel.

(e) equations of the given planes are $4x + 8y + z - 8 = 0$ $(a_1x + b_1y + c_1z + d_1 = 0)$ and $y + z - 4 = 0$ i.e., $0x + y + z - 4 = 0$ $(a_2x + b_2y + c_2z + d_2 = 0)$

Here, $\frac{a_1}{a_2} = \frac{4}{0}, \frac{b_1}{b_2} = \frac{8}{1}, \frac{c_1}{c_2} = \frac{1}{1}$

Since $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

Therefore, the given two planes are not parallel.

Again $a_1a_2 + b_1b_2 + c_1c_2$ = 4 x 0 + 8 x 1 + 1 x 1 = 0 + 8 + 1 = 9

Since $a_1a_2 + b_1b_2 + c_1c_2 \neq 0$

Therefore, the given two planes are not perpendicular.

Now let θ be the angle between the two planes.

$$\therefore \cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\cdot\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$= \dfrac{|4(0)+8(1)+1(1)|}{\sqrt{(4)^2+(8)^2+(1)^2}.\sqrt{(0)^2+(1)^2+(1)^2}}$

$= \dfrac{|8+1|}{\sqrt{81}.\sqrt{2}} = \dfrac{|9|}{9.\sqrt{2}} = \dfrac{1}{\sqrt{2}} = \cos 45^\circ$

$\Rightarrow \quad \theta = 45^\circ$

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14. In the following cases find the distances of each of the given points from the corresponding given plane:

(a) Point (0, 0, 0)

Plane $3x-4y+12z=3$

(b) Point $(3,-2,1)$

Plane $2x-y+2z+3=0$

(c) Point $(2,3,-5)$

Plane $x+2y-2z=9$

(d) Point $(-6,0,0)$

Plane $2x-3y+6z-2=0$

Ans. (a) Distance (of course perpendicular) of the point (0, 0, 0) from the plane $3x-4y+12z=3 \;\Rightarrow\; 3x-4y+12z-3=0$ is

$\dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

$= \dfrac{|3(0)-4(0)+12(0)-3|}{\sqrt{(3)^2+(-4)^2+(12)^2}}$

$= \frac{|3|}{\sqrt{9+16+144}} = \frac{3}{\sqrt{169}} = \frac{3}{13}$

(b) Length of perpendicular from the point $(3, -2, 1)$ on the plane $2x - y + 2z + 3 = 0$ is

$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$

$= \frac{|2(3) - (-2) + 2(1) + 3|}{\sqrt{(2)^2 + (-1)^2 + (2)^2}}$

$= \frac{|13|}{\sqrt{4+1+4}} = \frac{13}{\sqrt{9}} = \frac{13}{3}$

(c) Length of perpendicular from the point $(2, 3, -5)$ on the plane $x + 2y - 2z = 9 \Rightarrow x + 2y - 2z - 9 = 0$ is

$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$

$= \frac{|2(1) + 2(3) - 2(-5) - 9|}{\sqrt{(1)^2 + (2)^2 + (-2)^2}}$

$= \frac{|9|}{\sqrt{1+4+4}} = \frac{9}{\sqrt{9}} = \frac{9}{3} = 3$

(d) Length of perpendicular from the point $(-6, 0, 0)$ on the plane $2x - 3y + 6z - 2 = 0$ is

$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$

$= \frac{|2(-6) - 3(0) + 6(0) - 2|}{\sqrt{(2)^2 + (-3)^2 + (6)^2}}$

$= \frac{|-14|}{\sqrt{4+9+36}} = \frac{14}{\sqrt{49}} = \frac{14}{7} = 2$