CBSE Class–12 Mathematics

NCERT solution

Chapter - 10

Vector Algebra - Exercise 10.4

1. Find $\left|\vec{a}\times\vec{b}\right|$, if $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$.

Ans. Given: $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -7 & 7\\ 3 & -2 & 2\end{vmatrix}$$

Expanding along first row,

$$\vec{a}\times\vec{b}=\hat{i}\begin{vmatrix}-7 & 7\\ -2 & 2\end{vmatrix}-\hat{j}\begin{vmatrix}1 & 7\\ 3 & 2\end{vmatrix}+\hat{k}\begin{vmatrix}1 & -7\\ 3 & -2\end{vmatrix} = \hat{i}(-14+14)-\hat{j}(2-21)+\hat{k}(-2+21)$$

$= 0\hat{i}+19\hat{j}+19\hat{k}$

$$\therefore \left|\vec{a}\times\vec{b}\right|=\sqrt{(0)^2+(19)^2+(19)^2}=\sqrt{2(19)^2}=19\sqrt{2}$$

2. Find a unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ where $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$.

Ans. Given: $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$

On Adding $\vec{c}=\vec{a}+\vec{b}$ = $3\hat{i}+2\hat{j}+2\hat{k}$ + $\hat{i}+2\hat{j}-2\hat{k}$ = $4\hat{i}+4\hat{j}+0\hat{k}$

On Subtracting $\vec{d}=\vec{a}-\vec{b}$ = $3\hat{i}+2\hat{j}+2\hat{k}$ − $\hat{i}-2\hat{j}+2\hat{k}$ = $2\hat{i}+0\hat{j}+4\hat{k}$

Therefore, $\vec{n} = \vec{c} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix}$

Expanding along first row = $\hat{i}(16-0) - \hat{i}(16-0) + \hat{k}(0-8)$

$\Rightarrow \vec{n} = 16\hat{i} - 16\hat{j} - 8\hat{k}$

$\therefore |\vec{n}| = \sqrt{(16)^2 + (-16)^2 + (-8)^2}$ = $\sqrt{256+256+64} = \sqrt{576} = 24$

Therefore, a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is

$\hat{n} = \pm \frac{\vec{n}}{|\vec{n}|} = \pm \frac{(16\hat{i} - 16\hat{j} - 8\hat{k})}{24}$ = $\pm \left(\frac{16}{24}\hat{i} - \frac{16}{24}\hat{j} - \frac{8}{24}\hat{k} \right)$ = $\pm \left(\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k} \right)$

= $\pm \frac{2}{3}\hat{i} \mp \frac{2}{3}\hat{j} \mp \frac{2}{3}\hat{k}$

3. If a unit vector $\hat{a}$ makes an angle $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the components of $\hat{a}$.

Ans. Let $\hat{a} = x\hat{i} + y\hat{j} + z\hat{k}$ be a unit vector. ....(i)

$\Rightarrow |\hat{a}| = 1 \Rightarrow \sqrt{x^2 + y^2 + z^2} = 1$

Squaring both sides, $x^2 + y^2 + z^2 = 1$ ....(ii)

Given: Angle between vectors $\hat{a}$ and $\hat{i}$ is $\frac{\pi}{3}$.

$\therefore \cos\frac{\pi}{3} = \frac{\hat{a}.\hat{i}}{|\hat{a}|.|\hat{i}|} \Rightarrow \frac{1}{2} = \frac{x(1) + y(0) + z(0)}{(1)(1)}$

$\Rightarrow \frac{1}{2} = x$ ....(iii)

Again, given Angel between vectors $\hat{a}$ and $\hat{j}$ is $\frac{\pi}{4}$.

$\therefore \cos\frac{\pi}{4} = \frac{\hat{a}.\hat{j}}{|\hat{a}|.|\hat{j}|} \Rightarrow \frac{1}{\sqrt{2}} = \frac{x(0)+y(1)+z(0)}{(1)(1)}$

$\Rightarrow \frac{1}{\sqrt{2}} = y$ ....(iv)

Again, given Angel between vectors $\hat{a}$ and $\hat{k}$ is θ, where θ is acute angle.

$\therefore \cos\theta = \frac{\hat{a}.\hat{k}}{|\hat{a}|.|\hat{k}|} \Rightarrow \cos\theta = \frac{x(0)+y(0)+z(1)}{(1)(1)}$

$\Rightarrow \cos\theta = z$ ....(v)

Putting the values of x, y and z in eq. (ii),

$\frac{1}{4}+\frac{1}{2}+\cos^2\theta = 1 \Rightarrow \cos^2\theta = 1-\frac{1}{4}-\frac{1}{2}$

$\Rightarrow \cos^2\theta = \frac{4-1-2}{4} = \frac{1}{4} \Rightarrow \cos\theta = \pm\frac{1}{2}$

Since θ is acute angle, therefore $\cos\theta$ is positive and hence $\frac{1}{2} = \cos\frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3}$

From eq. (v), $z = \cos\theta = \frac{1}{2}$

Putting values of x, y and z in eq. (i), $\hat{a} = \frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}$

$\therefore$ Components of $\hat{a}$ are coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\hat{a}$

$\Rightarrow \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}$ and angle $\theta = \frac{\pi}{3}$

4. Show that $\left(\vec{a} - \vec{b}\right) \times \left(\vec{a} + \vec{b}\right) = 2\left(\vec{a} \times \vec{b}\right)$

Ans. L.H.S. = $\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)$ = $\vec{a}\times\vec{a}+\vec{a}\times\vec{b}-\vec{b}\times\vec{a}-\vec{b}\times\vec{b}$

$\left[\because \vec{a}\times\vec{a}=\vec{0}, \vec{b}\times\vec{b}=\vec{0}, \vec{b}\times\vec{a}=-\vec{a}\times\vec{b}\right]$

= $\vec{0}+\vec{a}\times\vec{b}+\vec{a}\times\vec{b}-\vec{0}$ = $2\left(\vec{a} \times \vec{b}\right)$ = R.H.S.

5. Find λ and μ if $\left(2\hat{i}+6\hat{j}+27\hat{k}\right)\times\left(\hat{i}+\lambda\hat{j}+\mu\hat{k}\right)=\vec{0}$.

Ans. Given: $\left(2\hat{i}+6\hat{j}+27\hat{k}\right)\times\left(\hat{i}+\lambda\hat{j}+\mu\hat{k}\right)=0$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix} = \vec{0}$$

Expanding along first row,

$\hat{i}(6\mu-27\lambda)-\hat{j}(2\mu-27)+\hat{k}(2\lambda-6)=\vec{0}$ = $0\hat{i}+0\hat{j}+0\hat{k}$

Comparing the coefficients of $\hat{i}, \hat{j}, \hat{k}$ on both sides, we have

$6\mu-27\lambda=0$ ....(i)

$2\mu-27=0$ ....(ii)

And $2\lambda-6=0$ ....(iii)

From eq. (ii), $2\mu-27=0 \Rightarrow \mu=\frac{27}{2}$

From eq. (iii), $2\lambda - 6 = 0 \Rightarrow \lambda = \frac{6}{2} = 3$

Putting the values of μ and λ in eq. (i),

$$6\left(\frac{27}{2}\right) - 27(3) = 0 \Rightarrow 3(27) - 27(3) = 0 \Rightarrow 0 = 0$$

Therefore, $\mu = \frac{27}{2}$ and $\lambda = 3$.

6. Given that $\vec{a}.\vec{b} = 0$ and $\vec{a} \times \vec{b} = \vec{0}$. What can you conclude about the vectors $\vec{a}$ and $\vec{b}$?

Ans. Given: $\vec{a}.\vec{b} = 0 \Rightarrow |\vec{a}|.|\vec{b}|\cos\theta = 0$

$\therefore |\vec{a}| = 0$ or $|\vec{b}| = 0$ or $\cos\theta = 0 \Rightarrow \theta = 90^\circ$

$\Rightarrow \vec{a} = 0$ or $\vec{b} = 0$ or vector $\vec{a}$ is perpendicular to $\vec{b}$. .....(i)

Again, given $\vec{a} \times \vec{b} = 0 \Rightarrow |\vec{a} \times \vec{b}| = 0 \Rightarrow |\vec{a}|.|\vec{b}|\sin\theta = 0$

$\therefore |\vec{a}| = 0$ or $|\vec{b}| = 0$ or $\sin\theta = 0 \Rightarrow \theta = 0^\circ$

$\Rightarrow \vec{a} = 0$ or $\vec{b} = 0$ or vector $\vec{a}$ and $\vec{b}$ are collinear or parallel. .....(ii)

Since, vectors $\vec{a}$ & $\vec{b}$ are perpendicular to each other as well as parallel are not possible. ..(iii)

Therefore, from eq. (i), (ii) and (iii), either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$

$\therefore \vec{a}.\vec{b} = 0$ and $\vec{a} \times \vec{b} = 0$

7. Let the vectors $\vec{a}, \vec{b}, \vec{c}$ be given as $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, b_1\hat{i} + b_2\hat{j} + b_3\hat{k}, c_1\hat{i} + c_2\hat{j} + c_3\hat{k},$

then show that $\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}.$

Ans. Given: Vector $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}, \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$

$\therefore\ \vec{b}+\vec{c}=\left(b_1+c_1\right)\hat{i}+\left(b_2+c_2\right)\hat{j}+\left(b_3+c_3\right)\hat{k}$

Now L.H.S. = $\vec{a}\times\left(\vec{b}+\vec{c}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1+c_1 & b_2+c_2 & b_3+c_3\end{vmatrix}$

= $\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$ + $\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\end{vmatrix}$

[By Property of Determinants]

= $\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ = R.H.S.

8. If either $\vec{a}=\vec{0}$ **and** $\vec{b}=\vec{0}$, **then** $\vec{a}\times\vec{b}=0.$ **Is the converse true? Justify your answer with an example.**

Ans. Given: Either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$

$\therefore\ \left|\vec{a}\right|=\left|\vec{0}\right|=0$ or $\left|\vec{b}\right|=\left|\vec{0}\right|=0$ ....(i)

$\therefore\ \left|\vec{a}\times\vec{b}\right|=\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta=0$

$\Rightarrow\ 0.\sin\theta=0$ [Using eq. (i)]

$\therefore\ \vec{a}\times\vec{b}=\vec{0}$ [By definition of zero vector]

But the converse is not true.

Let $\vec{a} = \hat{i} + \hat{j} + \hat{k} \Rightarrow \left|\vec{a}\right| = \sqrt{1+1+1} = \sqrt{3} \neq 0$

$\therefore$ $\vec{a}$ is a non-zero vector.

Let $\vec{b} = 2\left(\hat{i} + \hat{j} + \hat{k}\right) = 2\hat{i} + 2\hat{j} + 2\hat{k}$

$\Rightarrow \left|\vec{b}\right| = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3} \neq 0$

$\therefore$ $\vec{b}$ is a non-zero vector.

But $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{vmatrix}$

Taking 2 common from R_3 = $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \vec{0}$ [$\because$ R_2 and R_3 are identical]

9. Find the area of the triangle with vertices A (1, 1, 2), B (2, 3, 5) and C (1, 5, 5).

Ans. Vertices of ΔABC are A (1, 1, 2), B (2, 3, 5) and C (1, 5, 5).

$\therefore$ Position vector of point A = (1, 1, 2) = $\hat{i} + \hat{j} + 2\hat{k}$

Position vector of point B = (2, 3, 5) = $2\hat{i} + 3\hat{j} + 5\hat{k}$

Position vector of point C = (1, 5, 5) = $\hat{i} + 5\hat{j} + 5\hat{k}$

Now $\overline{AB}$ = Position vector of point B – Position vector of point A

= $2\hat{i} + 3\hat{j} + 5\hat{k} - \left(\hat{i} + \hat{j} + 2\hat{k}\right)$

= $2\hat{i} + 3\hat{j} + 5\hat{k} - \hat{i} - \hat{j} - 2\hat{k}$

= $\hat{i}+2\hat{j}+3\hat{k}$

And $\overrightarrow{AC}$ = Position vector of point C – Position vector of point A

= $\hat{i}+5\hat{j}+5\hat{k}-\left(\hat{i}+\hat{j}+2\hat{k}\right)$

= $\hat{i}+5\hat{j}+5\hat{k}-\hat{i}-\hat{j}-2\hat{k}$

= $0\hat{i}+4\hat{j}+3\hat{k}$

$\therefore \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix}$

= $\hat{i}(6-12)-\hat{j}(3-0)+\hat{k}(4-0) = -6\vec{i}-3\vec{j}+4\vec{k}$

Now Area of triangle ABC = $\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{1}{2}\sqrt{36+9+16}$

= $\frac{1}{2}\sqrt{61}$ sq. units

10. Find the area of the parallelogram whose adjacent sides are determined by the vectors $\vec{a}=\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}-7\hat{j}+\hat{k}$.

Ans. Given: Vectors representing two adjacent sides of a parallelogram are

$\vec{a}=\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}-7\hat{j}+\hat{k}$

$\therefore \vec{a}\times\vec{b}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix} = \hat{i}(-1+21)-\hat{j}(1-6)+\hat{k}(-7+2) = 20\hat{i}+5\hat{j}-5\hat{k}$

Now Area of parallelogram = $\left|\vec{a}\times\vec{b}\right|$

= $\sqrt{400+25+25} = \sqrt{450} = 15\sqrt{2}$ sq. units

11. Let the vectors $\vec{a}$ and $\vec{b}$ such that $|\vec{a}| = 3$, $|\vec{b}| = \frac{\sqrt{2}}{3}$, then $\vec{a} \times \vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is:

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$

(D) $\frac{\pi}{2}$

Ans. Given: $|\vec{a}| = 3$, $|\vec{b}| = \frac{\sqrt{2}}{3}$ and $\vec{a} \times \vec{b}$ is a unit vector.

$\Rightarrow |\vec{a} \times \vec{b}| = 1 \Rightarrow |\vec{a}|.|\vec{b}|\sin\theta = 1$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

$\Rightarrow 3\left(\frac{\sqrt{2}}{3}\right)\sin\theta = 1$

$\Rightarrow \sqrt{2}\sin\theta = 1$

$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \sin\theta = \sin\frac{\pi}{4}$

$\Rightarrow \theta = \frac{\pi}{4}$

Therefore, option (B) is correct.

12. Area of a rectangle having vertices A, B, C and D with position vectors

$-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k}, \hat{i}+\frac{1}{2}\hat{j}+4\hat{k}, \hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ **and** $-\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ **respectively is:**

(A) $\frac{1}{2}$

(B) 1

(C) 2

(D) 4

Ans. Given: ABCD is a rectangle.

Now $\overline{AB}$ = Position vector of point B – Position vector of point A

= $\hat{i}+\frac{1}{2}\hat{j}+4\hat{k}-\left(-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k}\right)$

= $\hat{i}+\frac{1}{2}\hat{j}+4\hat{k}+\hat{i}-\frac{1}{2}\hat{j}-4\hat{k}$

= $2\hat{i}+0\hat{j}+0\hat{k}$

$\therefore$ AB = $\left|\overline{AB}\right| = \sqrt{4+0+0} = \sqrt{4} = 2$

And $\overline{AD}$ = Position vector of point D – Position vector of point A

= $-\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}-\left(-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k}\right)$

= $-\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}+\hat{i}-\frac{1}{2}\hat{j}-4\hat{k}$

= $0\hat{i}-\hat{j}+0\hat{k}$

$\therefore$ AD = $\left|\overline{AD}\right| = \sqrt{0+1+0} = \sqrt{1} = 1$

$\therefore$ Area of rectangle ABCD = Length x Breadth = AB x AD = 2 x 1 = 2 sq. units

Therefore, option (C) is correct.