CBSE Class–12 Mathematics

NCERT solution

Chapter - 9

Differential Equations - Miscellaneous Exercise

1. For each of the differential equations given below, indicate its order and degree (if defined):

(i) $\frac{d^2y}{dx^2}+5x\left(\frac{dy}{dx}\right)^2-6y=\log x$

(ii) $\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y=\sin x$

(iii) $\frac{d^4y}{dx^4}-\sin\left(\frac{d^3y}{dx^3}\right)=0$

Ans. (i) Given: Differential equation $\frac{d^2y}{dx^2}+5x\left(\frac{dy}{dx}\right)^2-6y=\log x$

The highest order derivative present in this differential equation is $\frac{d^2y}{dx^2}$ and hence order of this differential equation if 2.

The given differential equation is a polynomial equation in derivatives and highest power of the highest order derivative $\frac{d^2y}{dx^2}$ is 1.

Therefore, Order = 2, Degree = 1

(ii) Given: Differential equation $\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y=\sin x$

The highest order derivative present in this differential equation is $\frac{dy}{dx}$ and hence order of this differential equation if 1.

The given differential equation is a polynomial equation in derivatives and highest power of the highest order derivative $\frac{dy}{dx}$ is 3.

Therefore, Order = 1, Degree = 3

(iii) Given: Differential equation $\frac{d^4y}{dx^4}-\sin\left(\frac{d^3y}{dx^3}\right)=0$

The highest order derivative present in this differential equation is $\frac{d^4y}{dx^4}$ and hence order of this differential equation if 4.

The given differential equation is not a polynomial equation in derivatives therefore, degree of this differential equation is not defined.

Therefore, Order = 4, Degree not defined

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2. For each of the exercises given below verify that the given function (implicit or explicit) is a solution of the corresponding differential equation:

(i) $xy=ae^x+be^{-x}+x^2 : x\frac{d^2y}{dx^2}+2\frac{dy}{dx}-xy+x^2-2=0$

(ii) $y=e^x\left(a\cos x+b\sin x\right): \frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0$

(iii) $y=x\sin 3x: \frac{d^2y}{dx^2}+9y-6\cos 3x=0$

(iv) $x^2=2y^2\log y: \left(x^2+y^2\right)\frac{dy}{dx}-xy=0$

Ans. (i) The given function is $xy = ae^{x} + be^{-x} + x^{2}$ ....(i)

To verify: Function (i) is a solution of D.E. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0$ ....(ii)

Differentiating both sides of eq. (i) w.r.t. x, $x\frac{dy}{dx} + y.1 = ae^{x} + be^{-x} + 2x$

Again differentiating both sides w.r.t. x,

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx}.1 + \frac{dy}{dx} = ae^{x} + be^{-x} + 2$$

$$\Rightarrow \quad x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = ae^{x} + be^{-x} + 2$$

Putting $ae^{x} + be^{-x} = xy - x^2$ from eq. (i), we have,

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy - x^2 + 2$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0$$

Therefore, Function given by eq. (i) is a solution of D.E. (ii).

(ii) The given function is $y = e^{x}\left(a\cos x + b\sin x\right)$ ....(i)

To verify: Function given by (i) is a solution of D.E. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ ....(ii)

From (i), $\frac{dy}{dx} = \frac{d}{dx}e^{x}\left(a\cos x + b\sin x\right) + e^{x}\frac{d}{dx}\left(a\cos x + b\sin x\right)$

$$\Rightarrow \quad \frac{dy}{dx} = e^{x}\left(a\cos x + b\sin x\right) + e^{x}\left(-a\sin x + b\cos x\right)$$

$\Rightarrow \dfrac{dy}{dx} = y + e^x\left(-a\sin x + b\cos x\right)$ [By eq. (i)] ....(iii)

$\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{dy}{dx} + e^x\left(-a\sin x + b\cos x\right) + e^x\left(-a\cos x - b\sin x\right)$

$\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{dy}{dx} + e^x\left(-a\sin x + b\cos x\right) - e^x\left(a\cos x + b\sin x\right)$

$\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{dy}{dx} + \left(\dfrac{dy}{dx} - y\right) - y$ [Using eq. (iii) and (i)]

$\Rightarrow \dfrac{d^2y}{dx^2} = 2\dfrac{dy}{dx} - 2y$

$\Rightarrow \dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$

Therefore, Function given by eq. (i) is a solution of D.E. (ii).

(iii) The given function is $y = x\sin 3x$ ....(i)

To verify: Function given by eq. (i) is a solution of D.E. $\dfrac{d^2y}{dx^2} + 9y - 6\cos 3x = 0$ ...(ii)

From eq. (i), $\dfrac{dy}{dx} = x\cos 3x.3 + \sin 3x.1$

$\Rightarrow \dfrac{dy}{dx} = 3x\cos 3x + \sin 3x$

$\Rightarrow \dfrac{d^2y}{dx^2} = 3\left[x(-\sin 3x)3 + \cos 3x.1\right] + (\cos 3x)3$

$\Rightarrow \dfrac{d^2y}{dx^2} = -9x\sin 3x + 6\cos 3x = -9y + 6\cos 3x$ [Using eq. (i)]

$$\Rightarrow \frac{d^2y}{dx^2}+9y-6\cos 3x=0$$

Therefore, Function given by eq. (i) is a solution of D.E. (ii).

(iv) The given function is $x^2 = 2y^2 \log y$ ....(i)

To verify: Function given by eq. (i) is a solution of D.E. $\left(x^2+y^2\right)\frac{dy}{dx}-xy=0$ ......(ii)

Differentiating both sides of eq. (i) w.r.t. x,

$$2x=2\left[y^2.\frac{1}{y}\frac{dy}{dx}+(\log y)2y\frac{dy}{dx}\right]$$

$$\Rightarrow x=\frac{dy}{dx}(y+2y\log y)$$

$$\Rightarrow \frac{dy}{dx}=\frac{x}{y+2y\log y}=\frac{x}{y(1+2\log y)}$$

Putting $2\log y=\frac{x^2}{y^2}$ from eq. (i), we get

$$\frac{dy}{dx}=\frac{x}{y\left(1+\frac{x^2}{y^2}\right)}=\frac{x}{y\left(\frac{y^2+x^2}{y^2}\right)}=\frac{xy^2}{y\left(x^2+y^2\right)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{xy}{x^2+y^2}$$

$$\Rightarrow \left(x^2+y^2\right)\frac{dy}{dx}=xy$$

$$\Rightarrow \left(x^2+y^2\right)\frac{dy}{dx}-xy=0$$

Therefore, Function given by eq. (i) is a solution of D.E. (ii).

3. Form the differential equation representing the family of curves $(x-a)^2+2y^2=a^2$, where a ia an arbitrary constant.

Ans. Equation of the given family of curves is $(x-a)^2+2y^2=a^2$

$\Rightarrow x^2+a^2-2ax+2y^2=a^2$

$\Rightarrow x^2-2ax+2y^2=0$

$\Rightarrow x^2+2y^2=2ax$ ....(i)

Here number of arbitrary constants is one only (a).

So, we will differentiate both sides of equation only once, w.r.t. x.

$\Rightarrow 2x+2.2y\frac{dy}{dx}=2a$

$\Rightarrow 2x+4y\frac{dy}{dx}=2a$ ....(ii)

Dividing eq. (i) by eq. (ii), we have

=> $\dfrac{x^2+2y^2}{2x+4y\frac{dy}{dx}}=\dfrac{2ax}{2a}$

$\Rightarrow \dfrac{x^2+2y^2}{2x+4y\frac{dy}{dx}}=x$

$\Rightarrow x\left(2x+4y\frac{dy}{dx}\right)=x^2+2y^2$

$$\Rightarrow 2x^2+4xy\frac{dy}{dx}=x^2+2y^2$$

$$\Rightarrow 4xy\frac{dy}{dx}=2y^2-x^2$$

$$\Rightarrow \frac{dy}{dx}=\frac{2y^2-x^2}{4xy}$$

4. Prove that $x^2-y^2=c\left(x^2+y^2\right)^2$ is the general equation of the differential equation $\left(x^3-3xy^2\right)\,dx=\left(y^3-3x^2y\right)\,dy$, where c is a parameter.

Ans. Given: Differential equation $\left(x^3-3xy^2\right)\,dx=\left(y^3-3x^2y\right)\,dy$ ....(i)

Here each coefficient of dx and dy is of same degree, i.e., 3, therefore differential equation looks to be homogeneous.

$$\therefore \frac{dy}{dx}=\frac{\left(x^3-3xy^2\right)}{\left(y^3-3x^2y\right)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1-3\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)^3-3\left(\frac{y}{x}\right)}=f\left(\frac{y}{x}\right) \quad \text{..........(ii)}$$

Therefore, the given differential equation is homogeneous.

Putting $\frac{y}{x}=v$,

$$\Rightarrow y=vx$$

$$\Rightarrow \frac{dy}{dx}=v.1+x\frac{dv}{dx}=v+x\frac{dv}{dx}$$

Putting these values in eq. (ii),

=> $v + x\frac{dv}{dx} = \frac{1-3v^2}{v^3-3v}$

$\Rightarrow x\frac{dv}{dx} = \frac{1-3v^2}{v^3-3v} - v = \frac{1-3v^2-v^4+3v^2}{v^3-3v}$

$\Rightarrow x\frac{dv}{dx} = \frac{1-v^4}{v^3-3v}$

$\Rightarrow x\left(v^3-3v\right)\, dv = \left(1-v^4\right)\, dx$

$\Rightarrow \frac{v^3-3v}{1-v^4}\, dv = \frac{dx}{x}$ [Separating variables]

Integrating both sides

, $\int \frac{v^3-3v}{1-v^4}\, dv = \int \frac{1}{x}\, dx = \log x + \log c$ ....(iii)

Now forming partial fraction of $\frac{v^3-3v}{1-v^4}$

$\frac{v^3-3v}{1-v^4} = \frac{v^3-3v}{\left(1-v^2\right)\left(1+v^2\right)}$

$\Rightarrow \frac{v^3-3v}{1-v^4} = \frac{v^3-3v}{(1-v)(1+v)\left(1+v^2\right)}$

$\Rightarrow \frac{v^3-3v}{1-v^4} = \frac{A}{(1-v)} = \frac{B}{(1+v)} = \frac{Cv+D}{\left(1+v^2\right)}$ ....(iv)

$\Rightarrow v^3-3v = A\,(1+v)\left(1+v^2\right) + B\,(1-v)\left(1+v^2\right) + (Cv+D)\left(1-v^2\right)$

$\Rightarrow v^3-3v = A\left(1+v^2+v+v^3\right) + B\left(1+v^2-v-v^3\right) + Cv - Cv^3 + D - Dv^2$

Comparing coefficients of like powers of

v^3 A – B – C = 1 ....(v)

v^2 A + B – D = 0 ....(vi)

v A – B + C = –3 ....(vii)

Constants A + B + D = 0 ....(viii)

Now eq. (v) – eq. (vii)

$\Rightarrow$ – 2C = 4

$\Rightarrow$ C = – 2

Eq. (vi) – eq. (viii)

$\Rightarrow$ – 2D = 0 $\Rightarrow$ D = 0

Putting C = – 2 in eq. (v), A – B + 2 = 1

$\Rightarrow$ A – B = –1 ....(ix)

Putting D = 0 in eq. (vi) A + B = 0 ....(x)

Adding eq. (ix) and (x) 2A = –1

$\Rightarrow$ A = $\frac{-1}{2}$

From eq. (x), B = –A = $\frac{1}{2}$

Putting the values of A, B, C and D in eq. (iv), we have

$$\frac{v^3-3v}{1-v^4}=\frac{\frac{-1}{2}}{1-v}+\frac{\frac{1}{2}}{1+v}-\frac{2v}{1+v^2}$$

$$\Rightarrow \int \frac{v^3-3v}{1-v^4}\,dv = \frac{-1}{2}\frac{\log(1-v)}{-1} + \frac{1}{2}\log(1+v) - \log(1+v^2)$$

$$\Rightarrow \int \frac{v^3-3v}{1-v^4}\,dv = \frac{1}{2}\log(1-v) + \frac{1}{2}\log(1+v) - \log(1+v^2)$$

$$\Rightarrow \int \frac{v^3-3v}{1-v^4}\,dv = \frac{1}{2}\left[\log(1-v) + \log(1+v)\right] - \log(1+v^2)$$

$$\Rightarrow \int \frac{v^3-3v}{1-v^4}\,dv = \frac{1}{2}\left[\log(1-v)(1+v)\right] - \log(1+v^2)$$

$$\Rightarrow \int \frac{v^3-3v}{1-v^4}\,dv = \log(1-v^2)^{\frac{1}{2}} - \log(1+v^2)$$

$$= \log\left(\frac{\sqrt{1-v^2}}{1+v^2}\right)$$

Putting this value in eq. (iii),

$$\Rightarrow \log\left(\frac{\sqrt{1-v^2}}{1+v^2}\right) = \log xc$$

$$\Rightarrow \frac{\sqrt{1-v^2}}{1+v^2} = xc$$

Squaring both sides and cross-multiplying,

$$\Rightarrow 1-v^2 = c^2x^2(1+v^2)^2$$

Putting $v = \frac{y}{x}$,

$$\Rightarrow 1-\frac{y^2}{x^2} = c^2x^2\left(1+\frac{y^2}{x^2}\right)^2$$

$\Rightarrow \dfrac{x^2-y^2}{x^2}=c^2x^2\dfrac{\left(x^2+y^2\right)^2}{x^4}$

$\Rightarrow \dfrac{x^2-y^2}{x^2}=c^2\dfrac{\left(x^2+y^2\right)^2}{x^2}$

$\Rightarrow x^2-y^2=\mathrm{C}\left(x^2+y^2\right)^2$ where $c^2=\mathrm{C}$

5. For the differential equation of the family of the circles in the first quadrant which touch the coordinate axes.

Ans. We know that the circle in the first quadrant which touches the co-ordinates axes has centre (a, a) where a is the radius of the circle.

$\therefore$ Equation of the circle is

$(x-a)^2+(y-a)^2=a^2$ ....(i)

$\Rightarrow x^2+y^2-2ax-2ay+a^2=0$

Differentiating with respect to x,

=> $2x+2yy'-2a-2ay'=0$

$\Rightarrow x+yy'-a-ay'=0$

$\Rightarrow x+yy'=a(1+y')$

$\Rightarrow a=\dfrac{x+yy'}{1+y'}$

Substituting value of a in eq. (i),

$\Rightarrow \left(x-\dfrac{x+yy'}{1+y'}\right)^2+\left(y-\dfrac{x+yy'}{1+y'}\right)^2=\left(\dfrac{x+yy'}{1+y'}\right)^2$

$$\Rightarrow \left(\frac{x+xy'-x+yy'}{1+y'}\right)^2+\left(\frac{y+yy'-x-yy'}{1+y'}\right)^2=\left(\frac{x+yy'}{1+y'}\right)^2$$

$$\Rightarrow (xy'-yy')^2+(y-x)^2=(x+yy')^2$$

$$\Rightarrow y'^2(x-y)^2+(x-y)^2=(x+yy')^2$$

$$\Rightarrow (x-y)^2(1+(y')^2)=(x+yy')^2$$

6. Find the general solution of the differential equation $\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$

Ans. Given: Differential Equation $\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$

$$\Rightarrow \frac{dy}{dx}=-\sqrt{\frac{1-y^2}{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\ dy=-\sqrt{1-y^2}\ dx$$

$$\Rightarrow \frac{dy}{\sqrt{1-y^2}}=\frac{-dx}{\sqrt{1-x^2}}$$

Integrating both sides

$$\Rightarrow \int\frac{1}{\sqrt{1-y^2}}\ dy=-\int\frac{1}{\sqrt{1-x^2}}\ dx$$

$\Rightarrow \sin^{-1}y = -\sin^{-1}x + c$

$$\Rightarrow \sin^{-1}x+\sin^{-1}y=c$$

7. Show that the general solution of the differential equation $\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1}=0$ **is**

given by (x + y + 1) = A (1 – x – y – 2xy), where A is parameter.

Ans. Given: Differential equation $\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1}=0$

$$\Rightarrow \frac{dy}{dx}=-\left(\frac{y^2+y+1}{x^2+x+1}\right)$$

$$\Rightarrow \frac{dy}{y^2+y+1}=\frac{-dx}{x^2+x+1}$$

$$\Rightarrow \frac{dy}{y^2+y+1}+\frac{dx}{x^2+x+1}=0$$

Integrating both sides,

$$\Rightarrow \int\frac{1}{y^2+y+1}\,dy+\int\frac{1}{x^2+x+1}\,dx=0 \quad \text{..........(i)}$$

Now $y^2+y+1=y^2+y+\frac{1}{4}-\frac{1}{4}+1$ [Completing the squares]

$$\Rightarrow y^2+y+1=\left(y+\frac{1}{2}\right)^2+\frac{3}{4}=\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2$$

Therefore, $\int\frac{1}{y^2+y+1}\,dy=\int\frac{1}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dy$

$$=\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}=\frac{2}{\sqrt{3}}\tan^{-1}\frac{2y+1}{\sqrt{3}}$$

Similarly, $\int \frac{1}{x^2+x+1}\,dx = \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}$

Putting these values in eq. (i),

$$\frac{2}{\sqrt{3}}\tan^{-1}\frac{2y+1}{\sqrt{3}}+\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}=c$$

$\Rightarrow \tan^{-1}\frac{2y+1}{\sqrt{3}}+\tan^{-1}\frac{2x+1}{\sqrt{3}}=\frac{\sqrt{3}}{2}c$ [Multiplying by $\frac{\sqrt{3}}{2}$]

$\Rightarrow \tan^{-1}\frac{\frac{2x+1}{\sqrt{3}}+\frac{2y+1}{\sqrt{3}}}{1-\frac{2x+1}{\sqrt{3}}.\frac{2y+1}{\sqrt{3}}}=\tan^{-1}c'$ where $c'=\frac{\sqrt{3}}{2}c$

$$\Rightarrow \frac{\sqrt{3}\left(2x+2y+2\right)}{3-\left(4xy+2x+2y+1\right)}=c'$$

$$\Rightarrow \sqrt{3}\left(2x+2y+2\right)=c'\left(2-2x-2y-4xy\right)$$

$$\Rightarrow 2\sqrt{3}\left(x+y+1\right)=2c'\left(1-x-y-2xy\right)$$

dividing by $2\sqrt{3}$

$$x+y+1=\frac{c'}{\sqrt{3}}\left(1-x-y-2xy\right)$$

$\Rightarrow x+y+1=\mathrm{A}\left(1-x-y-2xy\right)$ where A = $\frac{c'}{\sqrt{3}}$

8. Find the equation of the curve passing through the point $\left(0,\frac{\pi}{4}\right)$ whose differential equation is $\sin x\cos y\ dx+cox\sin y\ dy=0$

Ans. Given: Differential equation $\sin x\cos y\ dx+\cos x\sin y\ dy=0$

$\Rightarrow \sin x \cos y \; dx = -\cos x \sin y \; dy$

$\Rightarrow \dfrac{\sin x}{\cos x} \; dx = \dfrac{-\sin y}{\cos y} \; dy$

$\Rightarrow \tan x \; dx = -\tan y \; dy$

Integrating both sides,

$\Rightarrow \int \tan x \; dx = -\int \tan y \; dy$

$\Rightarrow \log|\sec x| = -\log|\sec y| + \log|c|$

$\Rightarrow \log|\sec x| + \log|\sec y| = \log|c|$

$\Rightarrow \log|\sec x \sec y| = \log|c|$

$\Rightarrow \sec x \sec y = c$ ....(i)

Now, curve (i) passes through $\left(0, \dfrac{\pi}{4}\right)$.

Therefore, putting $x = 0, y = \dfrac{\pi}{4}$ in eq. (i),

$\Rightarrow \sec 0 \sec \dfrac{\pi}{4} = c$

$\Rightarrow c = \sqrt{2}$

Putting $c = \sqrt{2}$ in eq. (i),

$\Rightarrow \sec x \sec y = \sqrt{2}$

$\Rightarrow \dfrac{\sec x}{\cos y} = \sqrt{2}$

$\Rightarrow \cos y = \dfrac{\sec x}{\sqrt{2}}$

9. Find the particular solution of the differential equation $\left(1+e^{2x}\right) dy + \left(1+y^2\right)e^x\, dx = 0$ given that $y=1$ when $x=0$.

Ans. Given: Differential equation $\left(1+e^{2x}\right) dy + \left(1+y^2\right)e^x\, dx = 0$

Dividing every term by $\left(1+y^2\right)\left(1+e^{2x}\right)$, we have

$\Rightarrow \dfrac{dy}{1+y^2} + \dfrac{e^x}{1+e^{2x}}\, dx = 0$

Integrating both sides,

$\Rightarrow \int \dfrac{1}{1+y^2}\, dy + \int \dfrac{e^x}{1+e^{2x}}\, dx = c$

$\Rightarrow \tan^{-1} y + \int \dfrac{e^x}{1+e^{2x}}\, dx = c$ ....(i)

Now to evaluate $\int \dfrac{e^x}{1+e^{2x}}\, dx$,

putting $e^x = t$

$\Rightarrow e^x = \dfrac{dt}{dx} \Rightarrow e^x\, dx = dt$

$\therefore \int \dfrac{e^x}{1+e^{2x}}\, dx = \int \dfrac{dt}{1+t^2} = \tan^{-1} t = \tan^{-1} e^x$

Putting this value in eq. (i), $\tan^{-1} y + \tan^{-1} e^x = c$ ....(ii)

Now putting $y=1, x=0$ in eq. (ii),

$\Rightarrow \tan^{-1} 1 + \tan^{-1} e^{0} = c$

$\Rightarrow \tan^{-1} 1 + \tan^{-1} 1 = c$

$\Rightarrow \frac{\pi}{4} + \frac{\pi}{4} = c$

$\Rightarrow c = \frac{\pi}{2}$

Putting $c = \frac{\pi}{2}$ in eq. (ii),

$\Rightarrow \tan^{-1} y + \tan^{-1} e^{x} = \frac{\pi}{2}$

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10. Solve the differential equation: $ye^{x/y}\ dx = \left(xe^{x/y} + y^{2}\right)\ dy\left(y \neq 0\right)$

Ans. Given: Differential equation $y.e^{x/y}\ dx = \left(x.e^{x/y} + y^{2}\right)\ dy,\ y \neq 0$

$\Rightarrow \frac{dx}{dy} = \frac{x.e^{x/y} + y^{2}}{y.e^{x/y}} = \frac{x.e^{x/y}}{y.e^{x/y}} + \frac{y^{2}}{y.e^{x/y}}$

$\Rightarrow \frac{dx}{dy} = \frac{x}{y} + y.e^{x/y}$ ....(i)

It is not a homogeneous differential equation because of presence of only y as a factor,

yet it can be solved by putting $\frac{x}{y} = v,$ i.e., $x = vy.$

$\Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$

Putting these values in eq. (i), we get

$\Rightarrow \quad v+y\dfrac{dv}{dy}=v+ye^{-v}$

$\Rightarrow \quad y\dfrac{dv}{dy}=y.e^{-v}$

$\Rightarrow \quad y\dfrac{dv}{dy}=\dfrac{y}{e^{v}}$

$\Rightarrow e^{v}dv=dy$

Integrate both sides, we get

$\Rightarrow \quad e^{v}=y+c$

$\Rightarrow \quad e^{x/y}=y+c$

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11. Find the particular solution of the differential equation $(x-y)(dx+dy)=dx-dy$ **given that** $y=-1$ **when** $x=0$.

Ans. Given: Differential equation $(x-y)(dx+dy)=dx-dy$

$\Rightarrow (x-y)dx+(x-y)dy=dx-dy$

$\Rightarrow (x-y)dx-dx+(x-y)dy+dy=0$

$\Rightarrow (x-y-1)dx+(x-y+1)dy=0$

$\Rightarrow (x-y-1)dx=-(x-y+1)dy$

$\Rightarrow \dfrac{dy}{dx}=-\dfrac{(x-y-1)}{x-y+1}$ ....(i)

Putting $x-y=t$

$\Rightarrow 1-\dfrac{dy}{dx}=\dfrac{dt}{dx}$

$$\Rightarrow -\frac{dy}{dx}=\frac{dt}{dx}-1$$

$$\Rightarrow \frac{dy}{dx}=\frac{-dt}{dx}+1$$

Putting this value in eq. (i),

$$\Rightarrow \frac{-dt}{dx}+1=-\left(\frac{t-1}{t+1}\right)$$

$$\Rightarrow \frac{-dt}{dx}=-1-\left(\frac{t-1}{t+1}\right)$$

$$\Rightarrow \frac{dt}{dx}=1+\left(\frac{t-1}{t+1}\right)=\frac{t+1+t-1}{t+1}$$

$$\Rightarrow \frac{dt}{dx}=\frac{2t}{t+1}$$

$$\Rightarrow (t+1)\ dt=2t\ dx$$

$$\Rightarrow \frac{t+1}{t}\ dt=2\ dx$$

Integrating both sides,

$$\Rightarrow \int\left(\frac{t+1}{t}\right)dt=2\int 1.dx$$

$$\Rightarrow \int\left(\frac{t}{t}+\frac{1}{t}\right)dt=2x+c$$

$$\Rightarrow \int\left(1+\frac{1}{t}\right)dt=2x+c$$

$$\Rightarrow t+\log|t|=2x+c$$

Putting $x-y=t$,

$\Rightarrow x-y+\log|x-y|=2x+c$

$\Rightarrow \log|x-y|=x+y+c$ ....(ii)

Now putting $y=-1, x=0$ in eq. (ii),

$\Rightarrow \log 1=0-1+c$

$\Rightarrow 0=-1+c$

$\Rightarrow c=1$

Putting $c=1$ in eq. (ii),

$\Rightarrow \log|x-y|=x+y+1$

12. Solve the differential equation: $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right)\frac{dx}{dy}=1\,(x\neq 0)$

Ans. Given: Differential equation $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right)\frac{dx}{dy}=1$

$\Rightarrow \frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}=\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}+\frac{y}{\sqrt{x}}=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

Comparing this equation with $\frac{dy}{dx}+\mathrm{P}y=\mathrm{Q}$,

P = $\frac{1}{\sqrt{x}}$ and Q = $\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

$$\int \mathrm{P}\ dx = \int \frac{1}{\sqrt{x}}\ dx = \int x^{-1/2}\ dx = \frac{x^{1/2}}{1/2} = 2\sqrt{x}$$

$\Rightarrow$ I.F. = $e^{\int \mathrm{P}\,dx} = e^{2\sqrt{x}}$

The general solution is

$\Rightarrow$ y (I.F.) = $\int \mathrm{Q}(\mathrm{I.F.})\ dx + c$

$\Rightarrow ye^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} e^{2\sqrt{x}} dx + c$

$\Rightarrow ye^{2\sqrt{x}} = \int \frac{1}{\sqrt{x}} dx + c$

$\Rightarrow ye^{2\sqrt{x}} = 2\sqrt{x} + c$

$\Rightarrow\ y = e^{-2\sqrt{x}}\left(2\sqrt{x} + c\right)$

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13. Find the particular solution of the differential equation $\frac{dy}{dx} + y\cot x = 4x\operatorname{cosec} x\,(x \neq 0)$ **given that** $y = 0$ **when** $x = \frac{\pi}{2}$.

Ans. Given: Differential equation $\frac{dy}{dx} + y\cot x = 4x\operatorname{cosec} x$

Comparing this equation with $\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$,

P = $\cot x$ and Q = $4x\operatorname{cosec} x$

$$\int \mathrm{P}\ dx = \int \cot x\ dx = \log \sin x$$

$\Rightarrow$ I.F. = $e^{\int \mathrm{P}\,dx} = e^{\log \sin x} = \sin x$

The general solution is

$\Rightarrow$ y (I.F.) = $\int \mathrm{Q}(\mathrm{I.F.})\ dx + c$

$\Rightarrow \ y(\sin x)=\int 4x\operatorname{cosec} x \sin x \, dx+c$

$\Rightarrow \ y(\sin x)=4\int x \frac{1}{\sin x} \sin x \, dx+c$

$\Rightarrow \ y(\sin x)=4\int x \, dx+c=4.\frac{x^2}{2}+c$

$\Rightarrow \ y\sin x=2x^2+c$ ....(i)

Now putting $y=0, x=\frac{\pi}{2}$ in eq. (i),

$\Rightarrow 0=2.\frac{\pi^2}{4}+c$

$\Rightarrow \ c=\frac{-\pi^2}{2}$

Putting $c=\frac{-\pi^2}{2}$ in eq. (i),

$\Rightarrow \ y\sin x=2x^2-\frac{\pi^2}{2}$

14. Find the particular solution of the differential equation $(x+1)\frac{dy}{dx}=2e^{-y}-1$ given that $y=0$ when $x=0$.

Ans. Given: Differential equation $(x+1)\frac{dy}{dx}=2e^{-y}-1$

$\Rightarrow (x+1)\frac{dy}{dx}=\frac{2}{e^y}-1=\frac{2-e^y}{e^y}$

$\Rightarrow (x+1)e^y \ dy=\left(2-e^y\right) \ dx$

$$\Rightarrow \frac{e^y\ dy}{2-e^y} = \frac{dx}{x+1}$$

Integrating both sides,

$$\Rightarrow \int \frac{e^y}{2-e^y}\ dy = \int \frac{1}{x+1}\ dx$$

Putting $e^y = t$

$$\Rightarrow e^y = \frac{dt}{dy}$$

$$\Rightarrow e^y\ dy = dt$$

$$\therefore \int \frac{dt}{2-t} = \log|x+1|$$

$$\Rightarrow \frac{\log|2-t|}{-1} = \log|x+1| + c$$

Putting $e^y = t$,

$$\Rightarrow -\log\left|2-e^y\right| = \log|x+1| + c$$

$$\Rightarrow \log|x+1| + \log\left|2-e^y\right| = -c$$

$$\Rightarrow \log|x+1|\left|2-e^y\right| = -c$$

$$\Rightarrow |x+1|\left|2-e^y\right| = e^{-c}$$

$$\Rightarrow (x+1)\left(2-e^y\right) = \pm e^{-c}$$

$\Rightarrow (x+1)\left(2-e^y\right) = C$ where C = $\pm e^{-c}$ ....(i)

Putting $x=0, y=0$ in eq. (i),

$\Rightarrow (1)(2-1)$ = C

$\Rightarrow$ C = 1

Putting C = 1 in eq. (i),

$\Rightarrow (x+1)(2-e^y)=1$

This solution may be written as

$\Rightarrow 2-e^y = \frac{1}{x+1}$

$\Rightarrow e^y = 2 - \frac{1}{x+1} = \frac{2x+1}{x+1}$

$\Rightarrow \log e^y = \log\left(\frac{2x+1}{x+1}\right)$

$\Rightarrow y = \log\left(\frac{2x+1}{x+1}\right)$

where expresses y as an explicit function of x.

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15. The population of a village increases continuously at the rate proportional to the number of its inhabitants present at any time. If the population of the village was 20,000 in 1999 and 25,000 in the year 2004, what will be the population of the village in 2009?

Ans. Let P be the population of the village at time t.

According to the question, Rate of increase of population of the village is proportional to the number of inhabitants.

$\Rightarrow \frac{dP}{dt} = kP$ where k > 0 because of increase and is the constant of proportionality

$\Rightarrow \quad \frac{dP}{P} = k\ dt$ [Separating variables]

Integrating both sides, $\int \frac{1}{P}\, dP = k\int 1\ dt$

$\Rightarrow \quad \log P = kt + c$ ....(i)

Now Population of the village was P = 20,000 in the year 1999.

Let us take the base year 1999 as $t = 0$.

Putting $t = 0$ and P = 20000 in eq. (i), $\log 20000 = c$

Now putting $\log 20000 = c$ in eq. (i), $\log P = kt + \log 20000$

$\Rightarrow \log P - \log 20000 = kt$

$\Rightarrow \quad \log \frac{P}{20000} = kt$ ....(ii)

Again Population of the village was P = 25,000 in the year 2004, when $t = 2004 - 1999 = 5$

Putting $t = 5$ and P = 25000 in eq. (ii), $\log \frac{25000}{20000} = 5k$

$\Rightarrow \quad 5k = \log \frac{5}{4}$

$\Rightarrow \quad k = \frac{1}{5}\log \frac{5}{4}$

Putting value of k in eq. (ii), $\log \frac{P}{20000} = \left(\frac{1}{5}\log \frac{5}{4}\right)t$ ....(iii)

To find the population in the year 2009, $t = 2009 - 1999 = 10$

Putting $t = 10$ in eq. (iii), $\log \frac{P}{20000} = \left(\frac{1}{5}\log \frac{5}{4}\right) \times 10$

$\Rightarrow \log \frac{P}{20000} = 2\log \frac{5}{4} = \log \left(\frac{5}{4}\right)^2 = \log \frac{25}{16}$

$=> P = \frac{25}{16} \times 20000$

$\Rightarrow$ 25 x 1250 = 31250

Choose the correct answer:

16. The general solution of the differential equation $\frac{y\ dx - x\ dy}{y} = 0$ **is:**

(A) xy = **C**

(B) $x = Cy^2$

(C) $y = Cx$

(D) $y = Cx^2$

Ans. Given: Differential equation $\frac{y\ dx - x\ dy}{y} = 0$

$\Rightarrow y\ dx - x\ dy = 0$

$\Rightarrow \frac{dx}{x} = \frac{dy}{y}$ [Separating variables]

Integrating both sides, $\log|x| = \log|y| + \log|c|$

$\Rightarrow \log|x| = \log|cy|$

$\Rightarrow x = \pm cy$

$\Rightarrow y = \pm \frac{1}{c} x$

$\Rightarrow \; y = Cx$ where C = $\pm\frac{1}{c}$

Therefore, option (C) is correct.

17. The general equation of a differential equation of the type $\frac{dy}{dx} + P_1 x = Q_1$ **is:**

(A) $ye^{\int P_1\, dy} = \int \left(Q_1 e^{\int P_1\, dy} \right) dy + C$

(B) $y.e^{\int P_1\, dx} = \int \left(Q_1 e^{\int P_1\, dx} \right) dx + C$

(C) $xe^{\int P_1\, dy} = \int \left(Q_1 e^{\int P_1\, dy} \right) dy + C$

(D) $xy.e^{\int P_1\, dx} = \int \left(Q_1 e^{\int P_1\, dx} \right) dx + C$

Ans. We know that general solution of differential equation of the type $\frac{dy}{dx} = P_1 x = Q_1$ is

$x.(\text{I.F.}) = \int Q_1 (\text{I.F.})\; dy + c$ where $(\text{I.F.}) = e^{\int P_1\, dy}$

$\therefore \; xe^{\int P_1\, dy} = \int \left(Q_1 e^{\int P_1\, dy} \right) dy + c$

Therefore, option (C) is correct.

18. The general solution of the differential equation $e^x\; dy + \left(ye^x + 2x \right)\; dx = 0$ **is:**

(A) $xe^y + x^2 = C$

(B) $xe^y + y^2 = C$

(C) $ye^x + x^2 = C$

(D) $ye^x + x^2 = C$

Ans. Given: Differential equation $e^x\ dy + \left(ye^x + 2x\right)\ dx = 0$

$$\Rightarrow\ e^x \frac{dy}{dx} + ye^x + 2x = 0$$

$$\Rightarrow\ e^x \frac{dy}{dx} + ye^x = -2x$$

$$\Rightarrow\ \frac{dy}{dx} + y = \frac{-2x}{e^x}$$

Comparing with $\frac{dy}{dx} = Px = Q$ P = 1 and Q = $\frac{-2x}{\frac{y}{x}}$

$\int P\ dx = \int 1\ dx = x$ I.F. = $e^{\int P\ dx} = e^x$

Solution is $y\left(\text{I.F.}\right) = \int Q\left(\text{I.F.}\right)\ dx + C$

$$\Rightarrow\ ye^x = \int \frac{-2x}{e^x} e^x\ dx + C$$

$$\Rightarrow\ ye^x = -2\int x\ dx + C$$

$$\Rightarrow\ ye^x = -2\frac{x^2}{2} + C$$

$$\Rightarrow\ ye^x = -x^2 + C$$

$$\Rightarrow\ ye^x + x^2 = C$$

Therefore, option (C) is correct.