CBSE Class–12 Mathematics

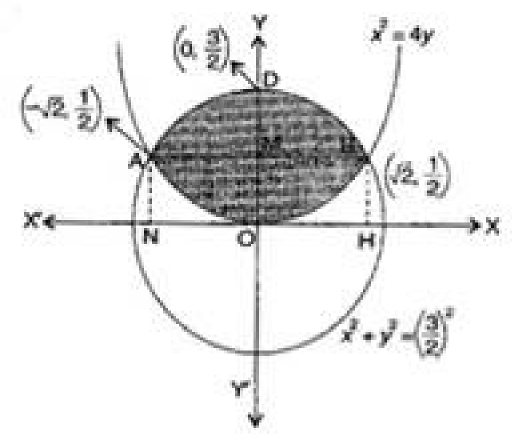
NCERT solution

Chapter - 8

Applications of Integrals - Exercise 8.2

1. Find the area of the circle $4x^2+4y^2=9$ which is interior to the parabola $x^2=4y$.

Ans. Step I. Equation of the circle is $4x^2+4y^2=9$



$\Rightarrow x^2+y^2=\frac{9}{4}$ ....(i)

Here centre is (0, 0) and radius is $\frac{3}{2}$

Equation of parabola is $x^2=4y$ .....(ii)

Step II. To find values of x and y.

Putting $x^2=4y$ in eq. (i), $4y+y^2=\frac{9}{4}$

$\Rightarrow 16y+4y^2=9$

$\Rightarrow 4y^2+16y-9=0$

$\Rightarrow 4y^2+18y-2y-9=0$

$\Rightarrow 2y(2y+9)-1(2y+9)=0$

$\Rightarrow (2y+9)(2y-1)=0$

$\Rightarrow 2y+9=0$ or $2y-1=0$

$\Rightarrow y=\frac{-9}{2}$ or $y=\frac{1}{2}$

Putting $y=\frac{-9}{2}$ in $x^2=4y$,

$\Rightarrow x^2=4\left(\frac{-9}{2}\right)=-18$

Putting $y=\frac{1}{2}$ in $x^2=4y$,

$\Rightarrow x^2=4\left(\frac{1}{2}\right)=2$

$\Rightarrow x=\pm\sqrt{2}$

$\therefore$ Points of intersections of circle (i) and parabola (ii) are A $\left(-\sqrt{2},\frac{1}{2}\right)$ and B $\left(\sqrt{2},\frac{1}{2}\right)$.

Step III. Area OBM = Area between parabola (ii) and $y-$axis

= $\left|\int_0^{\frac{1}{2}} x\,dy\right|$ $\left[\because \text{At O, } y=0 \text{ and at B, } y=\frac{1}{2}\right]$

= $\left|\int_0^{\frac{1}{2}} 2y^{\frac{1}{2}}\,dy\right|$ $\left[\because x^2=4y \Rightarrow x=2\sqrt{y}=2y^{\frac{1}{2}}\right]$

= $2.\dfrac{\left(y^{\frac{3}{2}}\right)_0^{\frac{1}{2}}}{\frac{3}{2}}$ = $2.\dfrac{2}{3}\left[\left(\dfrac{1}{2}\right)^{\frac{3}{2}}-0\right]$

= $\dfrac{4}{3}.\dfrac{1}{2}\sqrt{\dfrac{1}{2}}=\dfrac{\sqrt{2}}{3}$ ....(iii)

Step IV. Now area BDM = Area between circle (i) and $y-$axis

= $\left|\int_{\frac{1}{2}}^{\frac{3}{2}} x\,dy\right|$ $\left[\because \text{At B}, y=\dfrac{1}{2} \text{ and at D}, y=\dfrac{3}{2}\right]$

= $\left|\int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{\left(\frac{3}{2}\right)^2-y^2dy}\right|$ $\left[\because x^2=\left(\dfrac{3}{2}\right)^2-y^2 \Rightarrow x=\sqrt{\left(\dfrac{3}{2}\right)^2-y^2}\right]$

= $\left[\dfrac{y}{2}\sqrt{\left(\dfrac{3}{2}\right)^2-y^2}+\dfrac{\left(\frac{3}{2}\right)^2}{2}\sin^{-1}\dfrac{y}{\frac{3}{2}}\right]_{\frac{1}{2}}^{\frac{3}{2}}$

= $\dfrac{3}{4}\sqrt{\left(\dfrac{3}{2}\right)^2-\left(\dfrac{3}{2}\right)^2}+\dfrac{9}{8}\sin^{-1}\dfrac{\frac{3}{2}}{\frac{3}{2}}-\left[\dfrac{1}{4}\sqrt{\dfrac{9}{4}-\dfrac{1}{4}}+\dfrac{9}{8}\sin^{-1}\dfrac{\frac{1}{2}}{\frac{3}{2}}\right]$

= $\left(\dfrac{3}{4}\times 0\right)+\dfrac{9}{8}\sin^{-1}1-\left[\dfrac{1}{4}\sqrt{\dfrac{8}{4}}+\dfrac{9}{8}\sin^{-1}\dfrac{1}{3}\right]$

= $\dfrac{9}{8}\times\dfrac{\pi}{2}-\dfrac{1}{4}\sqrt{2}-\dfrac{9}{8}\sin^{-1}\dfrac{1}{3}$

= $\frac{9\pi}{16}-\frac{\sqrt{2}}{4}-\frac{9}{8}\sin^{-1}\frac{1}{3}$ ....(iv)

Step V. Required shaded area = Area AOBDA

= 2 (Area OBD) = 2 (Area OBM + Area MBD)

= $2\left[\frac{\sqrt{2}}{3}+\left(\frac{9\pi}{16}-\frac{\sqrt{2}}{4}-\frac{9}{8}\sin^{-1}\frac{1}{3}\right)\right]$ = $2\left[\sqrt{2}\left(\frac{1}{3}-\frac{1}{4}\right)+\frac{9\pi}{16}-\frac{9}{8}\sin^{-1}\frac{1}{3}\right]$

= $2\sqrt{2}\left(\frac{4-1}{12}\right)+\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}$

= $\frac{\sqrt{2}}{6}+\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}$ = $\frac{\sqrt{2}}{6}+\frac{9}{4}\left(\frac{\pi}{2}-\sin^{-1}\frac{1}{3}\right)$

= $\frac{\sqrt{2}}{6}+\frac{9}{4}\cos^{-1}\frac{1}{3}$

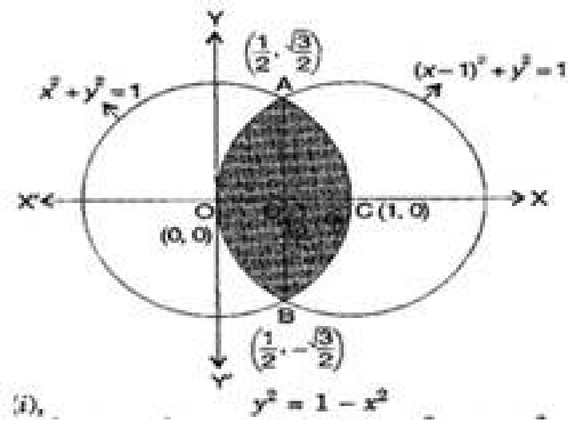
= $\frac{\sqrt{2}}{6}+\frac{9}{4}\sin^{-1}\sqrt{1-\frac{1}{9}}$ $\left[\text{Since,}\cos^{-1}x=\sin^{-1}\sqrt{1-x^2}\right]$

= $\frac{\sqrt{2}}{6}+\frac{9}{4}\sin^{-1}\sqrt{\frac{8}{9}}$

= $\frac{\sqrt{2}}{6}+\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$ sq. units

2. Find the area bounded by the curves $(x-1)^2+y^2=1$ **and** $x^2+y^2=1$.

Ans. Equations of two circles are



$x^2 + y^2 = 1$ ....(i)

And $(x-1)^2 + y^2 = 1$ ....(ii)

From eq. (i), $y^2 = 1 - x^2$

Putting this value in eq. (ii),

$(x-1)^2 + 1 - x^2 = 1$

$\Rightarrow x^2 + 1 - 2x + 1 - x^2 = 1$

$\Rightarrow -2x + 1 = 0$

$\Rightarrow x = \frac{1}{2}$

Putting $x = \frac{1}{2}$ in $y^2 = 1 - x^2$,

$y^2 = 1 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4} \Rightarrow y = \pm\frac{\sqrt{3}}{4}$

$\therefore$ The two points of intersections of circles (i) and (ii) are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$.

Now, from eq. (i), $y = \sqrt{1 - x^2}$ in first quadrant and from eq. (ii), $y = \sqrt{1 - (x-1)^2}$ in first quadrant.

$\therefore$ Required area OACBO = 2 x Area OAC = 2 (Area OAD + Area DAC)

$$= 2\left[\left|\int_0^{\frac{1}{2}} y \text{ of circle (ii) } dx + \left|\int_{\frac{1}{2}}^{1} y \text{ of circle (i) } dx\right|\right|\right]$$

$$= 2\left[\left|\int_0^{\frac{1}{2}} \sqrt{1-(x-1)^2}\ dx + \left|\int_{\frac{1}{2}}^{1} \sqrt{1-x^2}\ dx\right|\right|\right]$$

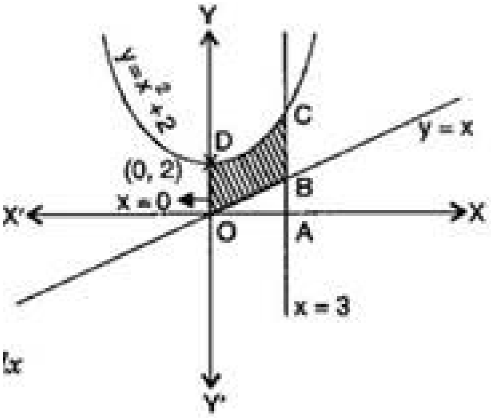
$$= 2\left[\left\{\frac{(x-1)\sqrt{1-(x-1)^2}}{2} + \frac{1}{2}\sin^{-1}(x-1)\right\}_0^{\frac{1}{2}} + \left\{\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right\}_{\frac{1}{2}}^{1}\right]$$

$$= \left\{-\frac{1}{2}\sqrt{\frac{3}{4}} + \sin^{-1}\left(-\frac{1}{2}\right)\right\} - \sin^{-1}(-1) - \left\{\frac{1}{2}\sqrt{\frac{3}{4}} + \sin^{-1}\frac{1}{2}\right\}$$

$$= -\frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{6} = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) \text{ sq. units}$$

3. Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ **and** $x = 3$.

Ans. Equation of the given curve is



$y = x^2 + 2$ ....(i)

$\Rightarrow x^2 = y - 2$

Here Vertex of the parabola is (0, 2).

Equation of the given line is $y = x.$ ....(ii)

Table of values for the line $y = x.$

x	0	1	2
y	0	1	2

We know that it is a straight line passing through the origin and having slope 1 i.e., making an angle of 45° with $x-$ axis.

Here also, Limits of integration area given to be $x = 0$ to $x = 3.$

$\therefore$ Area bounded by parabola (i) namely $y = x^2 + 2,$ the $x-$ axis and the ordinates $x = 0$ to $x = 3$ is the area OACD and $\int_0^3 y\, dx = \int_0^3 \left(x^2 + 2\right) dx$

= $\left(\frac{x^3}{3} + 2x\right)_0^3$ = (9 + 6) – 0 = 15 ....(iii)

Again Area bounded by parabola (ii) namely $y = x,$ the $x-$axis and the ordinates $x = 0$ to $x = 3$ is the area OAB and $\int_0^3 y\, dx = \int_0^3 x\, dx$

= $\left(\frac{x^2}{2}\right)_0^3$ = $\frac{9}{2}-0=\frac{9}{2}$ ....(iii)

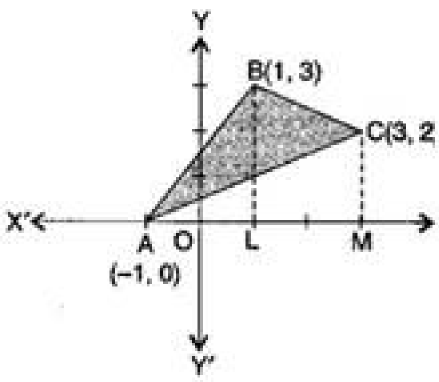
$\therefore$ Required area = Area OBCD = Area OACD – Area OAB

= Area given by eq. (iii) – Area given by eq. (iv)

= $15-\frac{9}{2}=\frac{21}{2}$ sq. units

4. Using integration, find the area of the region bounded by the triangle whose vertices are $(-1,0)$, (1, 3) and (3, 2).

Ans. Here, Vertices of triangle are A$(-1,0)$, B (1, 3) and



C (3, 2).

$\therefore$ Equation of the line is

$$y-0=\frac{3-0}{1-(-1)}(x-(-1))$$

$$\left[\because y-y_1=\frac{y_2-y_1}{x_2-x_1}(x_2-x_1)\right]$$

$$\Rightarrow\ y=\frac{3}{2}(x+1)$$

$\therefore$ Area of ΔABL = Area bounded by line AB and $x-$axis

= $\left|\int_{-1}^{1} y\ dx\right|$ $[\because \text{At A}, x=-1 \text{ and at B}, x=1]$

= $\left|\int_{-1}^{1} \frac{3}{2}(x+1)\ dx\right|$

= $\frac{3}{2}\left|\int_{-1}^{1}(x+1)\ dx\right|$ = $\frac{3}{2}\left|\left(\frac{x^2}{2}+x\right)_{-1}^{1}\right|$

= $\frac{3}{2}\left|\left(\frac{1}{2}+1\right)-\left(\frac{1}{2}-1\right)\right|$

= $\frac{3}{2}\left(\frac{3}{2}+\frac{1}{2}\right)$ = $\frac{3}{2}.\frac{4}{2}=3$ ....(i)

Again equation of line BC is $y-3=\frac{2-3}{3-1}(x-1) \Rightarrow y=\frac{1}{2}(7-x)$

$\therefore$ Area of trapezium BLMC = Area bounded by line BC and $x-$axis

= $\left|\int_{1}^{3} y\ dx\right|$ = $\left|\int_{1}^{3}\frac{1}{2}(7-x)\ dx\right|$

= $\frac{1}{2}\left|\left(7x-\frac{x^2}{2}\right)_{1}^{3}\right|$

= $\frac{1}{2}\left|\left(21-\frac{9}{2}\right)-\left(7-\frac{1}{2}\right)\right|$

= $\frac{1}{2}\left(21-\frac{9}{2}-7+\frac{1}{2}\right)$ = $\frac{1}{2}\left(\frac{42-9-14+1}{2}\right)$

= $\frac{1}{4}\times 20$ = 5 ....(ii)

Again equation of line AC is $y-0=\frac{2-0}{3-(-1)}\left(x-(-1)\right) \Rightarrow y=\frac{1}{2}(x+1)$

$\therefore$ Area of triangle ACM = Area bounded by line AC and $x-$ axis

= $\left|\int_{-1}^{3} y\ dx\right|$ = $\left|\int_{-1}^{3}\frac{1}{2}(x+1)\ dx\right|$ = $\frac{1}{2}\left|\left(\frac{x^2}{2}+x\right)_{-1}^{3}\right|$

= $\frac{1}{2}\left|\left(\frac{9}{2}+3\right)-\left(\frac{1}{2}-1\right)\right|$

= $\frac{1}{2}\left(\frac{9}{2}+3-\frac{1}{2}+1\right)$

= $\frac{1}{2}\left(\frac{9+6-1+2}{2}\right)$

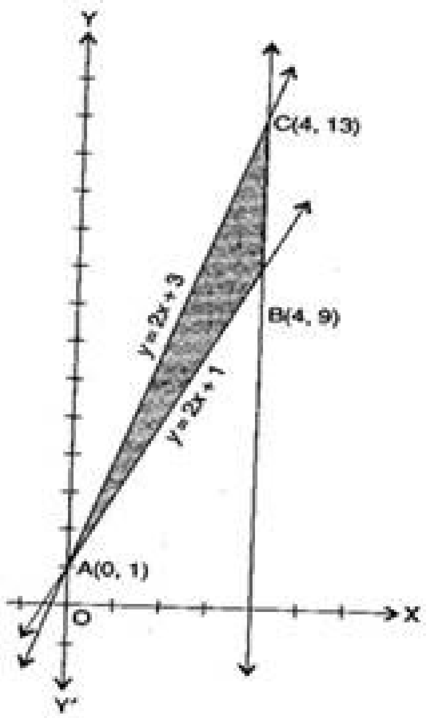
= $\frac{1}{4}\times 16$ = 4 ....(iii)

$\therefore$ Required area = Area of ΔABL + Area of Trapezium BLMC – Area of ΔACM

= 3 + 5 – 4 = 4 sq. units

5. Using integration, find the area of the triangular region whose sides have the equations $y=2x+1$, $y=3x+1$ and $x=4$.

Ans. Equations of one side of triangle is



$y = 2x+1$ ....(i)

second line of triangle is $y = 3x+1$ ....(ii)

third line of triangle is $x = 4$ ....(iii)

Solving eq. (i) and (ii), we get $x = 0$ and $y = 1$

$\therefore$ Point of intersection of lines (i) and (ii) is A (0, 1)

Putting $x = 4$ in eq. (i), we get $y = 9$

$\therefore$ Point of intersection of lines (i) and (iii) is B (4, 9)

Putting $x = 4$ in eq. (i), we get $y = 13$

$\therefore$ Point of intersection of lines (ii) and (iii) is C (4, 13)

$\therefore$ Area between line (ii) i.e., AC and $x-$ axis

$$= \left|\int_0^4 y\ dx\right| = \left|\int_0^4 (3x+1)\ dx\right| = \left(\frac{3x^2}{2} + x\right)_0^4$$

= 24 + 4 = 28 sq. units ....(iv)

Again Area between line (i) i.e., AB and $x-$ axis

= $\left|\int_0^4 y\ dx\right|$ = $\left|\int_0^4 (2x+1)\ dx\right|$ = $\left(x^2+x\right)_0^4$

= 16 + 4 = 20 sq. units ....(v)

Therefore, Required area of ΔABC

= Area given by (iv) – Area given by (v)

= 28 – 20 = 8 sq. units

6. Choose the correct answer:

Smaller area enclosed by the circle $x^2+y^2=4$ and the line $x+y=2$ is:

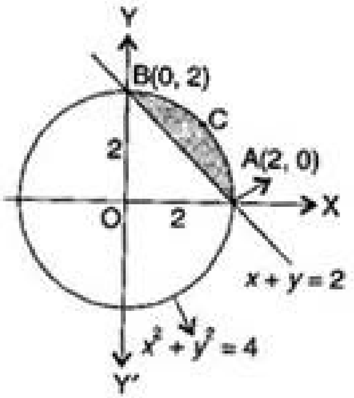
(A) $2(\pi-2)$

(B) $\pi-2$

(C) $2\pi-1$

(D) $2(\pi+2)$

Ans. Step I. Equation of circle is $x^2+y^2=2^2$ ....(i)



$\Rightarrow\ y=\sqrt{2^2-x^2}$ ....(ii)

Also, equation of the line is $x+y=2$ ....(iii)

Table of values

x	0	2
y	2	0

Therefore graph of equation (iii) is the straight line joining the points (0, 2) and (2, 0).

Step II. From the graph of circle (i) and straight line (iii), it is clear that points of intersections of circle (i) and straight line (iii) are A (2, 0) and B (0, 2).

Step III. Area OACB, bounded by circle (i) and coordinate axes in first quadrant = $\left|\int_0^2 y\ dx\right|$ =

$$\left|\int_0^2 \sqrt{2^2-x^2}\ dx\right|$$

$$= \left(\frac{x}{2}\sqrt{2^2-x^2}+\frac{2^2}{2}\sin^{-1}\frac{x}{2}\right)_0^2$$

$$= \left(\frac{2}{2}\sqrt{4-4}+2\sin^{-1}1\right)-\left(0+2\sin^{-1}0\right)$$

$= 0+2\left(\frac{\pi}{2}\right)-2(0)=\pi$ sq. units ....(iv)

Step IV. Area of triangle OAB, bounded by straight line (iii) and coordinate axes

$$= \left|\int_0^2 y\ dx\right| = \left|\int_0^2 (2-x)\ dx\right|$$

$$= \left(2x-\frac{x^2}{2}\right)_0^2$$

= $(4-2)-(0-0)$ = 2 sq. units ....(v)

Step V. Required shaded area = Area OACB given by (iv) – Area of triangle OAB by (v)

= $(\pi-2)$ sq. units

Therefore, option (B) is correct.

7. **Choose the correct answer:**

Area lying between the curves $y^2=4x$ and $y=2x$ is:

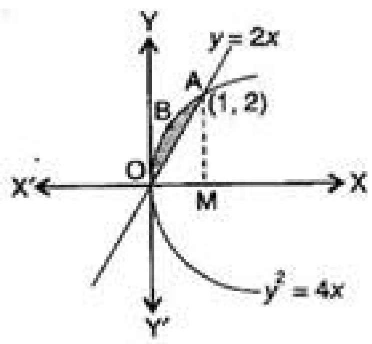
(A) $\frac{2}{3}$

(B) $\frac{1}{3}$

(C) $\frac{1}{4}$

(D) $\frac{3}{4}$

Ans. Equation of curve (parabola) is $y^2=4x$ .....(i)



$\Rightarrow$ $y=2\sqrt{x}$ = $2x^{\frac{1}{2}}$ .....(ii)

Equation of another curve (line) is $y=2x$ .....(iii)

Solving eq. (i) and (iii), we get $x=0$ or $x=1$ and $y=0$ or $y=2$

Therefore, Points of intersections of circle (i) and line (ii) are O (0, 0) and A (1, 2).

Now Area OBAM = Area bounded by parabola (i) and $x-$ axis = $\left|\int_0^1 y\ dx\right|$ = $\left|\int_0^1 2x^{\frac{1}{2}}\ dx\right|$ =

$$2\frac{\left(x^{\frac{3}{2}}\right)_0^1}{\frac{3}{2}}$$

= $\frac{4}{3}(1-0)=\frac{4}{3}$ ....(iv)

Also, Area ΔOAM = Area bounded by parabola (iii) and $x-$ axis

= $\left|\int_0^1 y\ dx\right|$ = $\left|\int_0^1 2x\ dx\right|$ = $2\left(\frac{x^2}{2}\right)_0^1$

= $(1-0)=1$ ....(v)

Now Required shaded area OBA = Area OBAM – Area of ΔOAM

= $\frac{4}{3}-1=\frac{4-3}{3}=\frac{1}{3}$ sq. units

Therefore, option (B) is correct.