CBSE Class–12 Mathematics

NCERT solution

Chapter - 7

Integrals - Exercise 7.10

Evaluate the integrals in Exercises 1 to 8 using substitutions.

1. $\int_0^1 \frac{x}{x^2+1}\, dx$

Ans. Let I = $\int_0^1 \frac{x}{x^2+1}\, dx$

= $\frac{1}{2}\int_0^1 \frac{2x}{x^2+1}\, dx$ ....(i)

Putting $x^2+1=t$

$\Rightarrow\ 2x=\frac{dt}{dx}$

$\Rightarrow 2x\, dx=dt$

To change the limits of integration from x to t

when x = 0, t = x^2 +1 = 0 +1 = 1

when x =1, t = x^2 +1 = 1 +1 = 2

$\therefore$ From eq. (i),

$=\frac{1}{2}\int_1^2 \frac{dt}{t}$

= $\frac{1}{2}\left(\log|t|\right)_1^2$

= $\frac{1}{2}\left(\log|2| - \log|1|\right)$

= $\frac{1}{2}\left(\log 2 - \log 1\right)$

= $\frac{1}{2}\left(\log 2 - 0\right)$

= $\frac{1}{2}\log 2$ Ans.

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2. $\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\, d\phi$

Ans. Let I = $\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\, d\phi$ ....(i)

Putting $\sin\phi = t$

$\Rightarrow \cos\phi = \frac{dt}{d\phi}$

$\Rightarrow \cos\phi\, d\phi = dt$

To change the limits of integration from ϕ to t

When $\phi = 0$, $t = \sin\phi = \sin 0^\circ = 0$

When $\phi = \frac{\pi}{2}$, $t = \sin\phi = \sin\frac{\pi}{2} = 1$

$\therefore$ From eq. (i),

$$I = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^4\phi\cos\phi\, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\left(\cos^2\phi\right)^2 \cos\phi\, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\left(1-\sin^2\phi\right)^2 \cos\phi\, d\phi$$

$$= \int_0^1 \sqrt{t}\left(1-t^2\right)^2 dt$$

$$= \int_0^1 t^{\frac{1}{2}}\left(1+t^4-2t^2\right) dt$$

$$= \int_0^1 \left(t^{\frac{1}{2}}+t^{\frac{1}{2}+4}-2t^{\frac{1}{2}+2}\right) dt$$

$$= \int_0^1 \left(t^{\frac{1}{2}}+t^{\frac{9}{2}}-2t^{\frac{5}{2}}\right) dt$$

$$= \int_0^1 t^{\frac{1}{2}}dt+\int_0^1 t^{\frac{9}{2}}dt-2\int_0^1 t^{\frac{5}{2}}dt$$

$$= \frac{2}{3}\left(t^{\frac{3}{2}}\right)_0^1+\frac{2}{11}\left(t^{\frac{11}{2}}\right)_0^1-\frac{4}{7}\left(t^{\frac{7}{2}}\right)_0^1$$

$$= \frac{2}{3}(1-0)+\frac{2}{11}(1-0)-\frac{4}{7}(1-0)$$

$$= \frac{2}{3}+\frac{2}{11}-\frac{4}{7}$$

= $\dfrac{154+42+132}{231}$

= $\dfrac{64}{231}$ Ans.

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3. $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$

Ans. Let I = $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\ dx\right)$ ....(i)

Putting $x = \tan\theta$

$\Rightarrow \frac{dx}{d\theta}\sec^2\theta$

$\Rightarrow dx = \sec^2\theta\ d\theta$

Limits of integration, when $x = 0, \tan\theta = 0 = \tan 0° \Rightarrow \theta = 0$

when $x = 1, \tan\theta = 1 = \tan\frac{\pi}{4}$

$\Rightarrow \theta = \frac{\pi}{4}$

$\therefore$ From eq. (i),

I = $\int_0^{\frac{\pi}{4}}\left(\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\right)\sec^2\theta\ d\theta$

= $\int_0^{\frac{\pi}{4}}\left(\sin^{-1}(\sin 2\theta)\right)\sec^2\theta\ d\theta$

$= \int_0^{\frac{\pi}{4}} 2\theta \sec^2 \theta \, d\theta$

$= 2\int_0^{\frac{\pi}{4}} \theta \sec^2 \theta \, d\theta$

[Applying Product Rule]

$= 2\left[(\theta.\tan\theta)_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 1.\tan\theta \, d\theta \right]$

$= 2\left[\frac{\pi}{4}\tan\frac{\pi}{4} - 0 - \int_0^{\frac{\pi}{4}} \tan\theta \, d\theta \right]$

$= 2\left[\frac{\pi}{4} - (\log\sec\theta)_0^{\frac{\pi}{4}} \right]$

$= 2\left[\frac{\pi}{4} - \left(\log\sec\frac{\pi}{4} - \log\sec 0° \right) \right]$

$= 2\left[\frac{\pi}{4} - \left(\log\sqrt{2} - \log 1 \right) \right]$

$= \frac{\pi}{2} - 2\log 2^{\frac{1}{2}}$

$= \frac{\pi}{2} - 2.\frac{1}{2}\log 2$

$= \frac{\pi}{2} - \log 2$ Ans.

4. $\int_0^2 x\sqrt{x+2}\ dx$

Ans. Let I = $\int_0^2 x\sqrt{x+2}\ dx$ ....(i)

Putting $\sqrt{x+2}=t$

$\Rightarrow\ x+2=t^2$

$\Rightarrow\ \frac{dx}{dt}=2t$

$\Rightarrow\ dx=2t\ dt$

Limits of integration when $x=0,\ t=\sqrt{2}$ and when $x=2,\ t=\sqrt{4}=2$

$\therefore$ From eq. (i),

I = $\int_{\sqrt{2}}^{2}\left(t^2-2\right)t.2t\ dt$

= $2\int_{\sqrt{2}}^{2} t^2\left(t^2-2\right)\ dt$

= $2\int_{\sqrt{2}}^{2}\left(t^4-2t^2\right)\ dt$

= $2\left[\left(\frac{t^5}{5}\right)_{\sqrt{2}}^{2}-2\left(\frac{t^3}{3}\right)_{\sqrt{2}}^{2}\right]$

= $2\left[\frac{1}{5}\left(2^5-\left(\sqrt{2}\right)^5\right)-\frac{2}{3}\left(2^3-\left(\sqrt{2}\right)^3\right)\right]$

= $2\left[\frac{1}{5}\left(32-4\sqrt{2}\right)-\frac{2}{3}\left(8-2\sqrt{2}\right)\right]$

= $2\left[\frac{32}{5}-\frac{4\sqrt{2}}{5}-\frac{16}{3}+\frac{4\sqrt{2}}{3}\right]$

= $2\left[\frac{96-12\sqrt{2}-80+20\sqrt{2}}{15}\right]$

= $\frac{2}{15}\left(16+8\sqrt{2}\right)$

= $\frac{16\sqrt{2}}{15}\left(\sqrt{2}+1\right)$ Ans.

5. $\int_0^{\frac{\pi}{2}}\frac{\sin x}{1+\cos^2 x}\,dx$

Ans. Let I = $\int_0^{\frac{\pi}{2}}\frac{\sin x}{1+\cos^2 x}\,dx$

= $-\int_0^{\frac{\pi}{2}}\frac{-\sin x}{1+\cos^2 x}\,dx$ ....(i)

Putting $\cos x = t$

$\Rightarrow\ -\sin x = \frac{dt}{dx}$

$\Rightarrow\ -\sin x\,dx = dt$

Limits of integration when $x=0, t=\cos 0^\circ = 1$ and when $x=\frac{\pi}{2}, t=\cos\frac{\pi}{2}=0$

$\therefore$ From eq. (i), I = $-\int_1^0 \frac{dt}{1+t^2}$

= $-\int_1^0 \frac{1}{1+t^2}\ dt$

= $-\left(\tan^{-1} t\right)_0^1$

= $-\left(\tan^{-1} 0 - \tan^{-1} 1\right)$

= $-\left(0-\frac{\pi}{4}\right)=\frac{\pi}{4}$ Ans.

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6. $\int_0^2 \frac{dx}{x+4-x^2}\ dx$

Ans. $\int_0^2 \frac{dx}{x+4-x^2}$

= $\int_0^2 \frac{dx}{-x^2+x+4}$

= $\int_0^2 \frac{dx}{-\left(x^2-x-4\right)}$

= $\int_0^2 \frac{dx}{-\left(x^2-x+\frac{1}{4}-\frac{1}{4}-4\right)}$

= $\int_0^2 \frac{dx}{-\left[\left(x-\frac{1}{2}\right)^2-\frac{17}{4}\right]}$

$$= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}$$

$$= \frac{1}{2 \times \frac{\sqrt{17}}{2}} \left[\log \left| \frac{\frac{\sqrt{17}}{2} + \left(x - \frac{1}{2}\right)}{\frac{\sqrt{17}}{2} - \left(x - \frac{1}{2}\right)} \right| \right]_0^2$$

$$\left[\because \int \frac{1}{a^2 - x^2}\, dx = \frac{1}{2a} \log \left| \frac{a + x}{a - x} \right| \right]$$

$$= \frac{1}{\sqrt{17}} \left[\log \left| \frac{\sqrt{17} + 2x - 1}{\sqrt{17} - 2x + 1} \right| \right]_0^2$$

$$= \frac{1}{\sqrt{17}} \left[\log \left| \frac{\sqrt{17} + 3}{\sqrt{17} - 3} \right| - \log \left| \frac{\sqrt{17} - 1}{\sqrt{17} + 1} \right| \right]$$

$$= \frac{1}{\sqrt{17}} \log \left(\frac{\sqrt{17}+3}{\sqrt{17}-3} \times \frac{\sqrt{17}+1}{\sqrt{17}-1} \right)$$

$$= \frac{1}{\sqrt{17}} \log \left(\frac{20 + 4\sqrt{17}}{20 - 4\sqrt{17}} \right)$$

$$= \frac{1}{\sqrt{17}} \log \frac{4\left(5 + \sqrt{17}\right)}{4\left(5 - \sqrt{17}\right)}$$

$$= \frac{1}{\sqrt{17}} \log \frac{\left(5 + \sqrt{17}\right)}{\left(5 - \sqrt{17}\right)}$$

= $\frac{1}{\sqrt{17}}\log\left(\frac{5+\sqrt{17}}{5-\sqrt{17}}\times\frac{5+\sqrt{17}}{5+\sqrt{17}}\right)$

= $\frac{1}{\sqrt{17}}\log\frac{\left(5+\sqrt{17}\right)^2}{25-17}$

= $\frac{1}{\sqrt{17}}\log\frac{42+10\sqrt{17}}{8}$

= $\frac{1}{\sqrt{17}}\log\frac{21+5\sqrt{17}}{4}$ Ans.

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7. $\int_{-1}^{1}\frac{dx}{x^2+2x+5}$

Ans. Let I = $\int_{-1}^{1}\frac{dx}{x^2+2x+5}$

= $\int_{-1}^{1}\frac{dx}{x^2+2x+1+4}$

= $\int_{-1}^{1}\frac{dx}{(x+1)^2+2^2}$ ....(i)

Putting $x+1=t$

$\Rightarrow\ 1=\frac{dt}{dx}$

$\Rightarrow\ dx=dt$

Limits of integration when $x=-1, t=-1+1=0$ and when $x=1, t=1+1=2$

$\therefore$ From eq. (i), I = $\int_0^2 \frac{1}{t^2+2^2}\, dt$

= $\frac{1}{2}\left(\tan^{-1}\frac{t}{2}\right)_0^2$

$\left[\because \int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$

= $\frac{1}{2}\left[\tan^{-1}\frac{2}{2} - \tan^{-1}\frac{0}{2}\right]$

= $\frac{1}{2}\left[\tan^{-1}1 - \tan^{-1}0\right]$

= $\frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{8}$ Ans.

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8. $\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x}\, dx$

Ans. Let I = $\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x}\, dx$ ....(i)

Putting $2x = t$

$\Rightarrow 2 = \frac{dt}{dx}$

$\Rightarrow 2\, dx = dt$

$\Rightarrow dx = \frac{dt}{2}$

Limits of integration when $x = 1, t = 2\times 1 = 2$ and when $x = 2, t = 2\times 2 = 4$

$\therefore$ From eq. (i),

$$I = \int_2^4 \left(\frac{1}{\frac{t}{2}} - \frac{1}{2\left(\frac{t}{2}\right)^2} \right) e^t \frac{dt}{2}$$

$$= \int_2^4 \left(\frac{2}{t} - \frac{2}{t^2} \right) e^t \frac{dt}{2}$$

$$= \int_2^4 \frac{1}{2} . 2 \left(\frac{1}{t} - \frac{1}{t^2} \right) e^t \; dt$$

$$= \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2} \right) e^t \; dt$$

$$= \int_2^4 \{ f(t) + f'(t) \} e^t \; dt$$

$$= \{ e^t f(t) \}_2^4$$

$$= \left(\frac{e^t}{t} \right)_2^4$$

$$= \frac{e^4}{4} - \frac{e^2}{2}$$

$$= \frac{e^4 - 2e^2}{4}$$

$$= \frac{e^2 (e^2 - 2)}{4} \text{ Ans.}$$

Choose the correct answer in Exercises 9 and 10.

9. The value of the integral $\int_{\frac{1}{3}}^{1} \frac{\left(x-x^3\right)^{\frac{1}{3}}}{x^4}\,dx$ **is:**

(A) 6

(B) 0

(C) 3

(D) 4

Ans. Let I = $\int_{\frac{1}{3}}^{1} \frac{\left(x-x^3\right)^{\frac{1}{3}}}{x^4}\,dx$

= $\int_{\frac{1}{3}}^{1} \frac{\left[x^3\left(\frac{x}{x^3}-1\right)\right]^{\frac{1}{3}}}{x^4}\,dx$

= $\int_{\frac{1}{3}}^{1} \frac{x\left(x^{-2}-1\right)^{\frac{1}{3}}}{x^4}\,dx$

= $\int_{\frac{1}{3}}^{1} \left(x^{-2}-1\right)^{\frac{1}{3}} x^{-3}\,dx$

= $\frac{-1}{2}\int_{\frac{1}{3}}^{1} \left(x^{-2}-1\right)^{\frac{1}{3}}\left(-2x^{-3}\right)\,dx$..........(i)

Putting $x^{-2}-1=t$

$\Rightarrow -2x^{-3}=\frac{dt}{dx}$

$\Rightarrow -2x^{-3}\ dx = dt$

Limits of integration when

$x=\frac{1}{3}, t=\left(\frac{1}{3}\right)^{-2}-1=3^2-1=8$ and when $x=1, t=(1)^{-2}-1=1-1=0$

$\therefore$ From eq. (i),

I = $\frac{-1}{2}\int_8^0 t^{\frac{1}{3}}\ dt$

= $\frac{-1}{2}\left(\frac{t^{\frac{4}{3}}}{\frac{4}{3}}\right)_8^0$

= $\frac{-1}{2}.\frac{3}{4}\left[0-8^{\frac{4}{3}}\right]$

= $\frac{-3}{8}\left[-(2)^{3\times\frac{4}{3}}\right]$

= $\frac{-3}{8}\left(-2^4\right)$

= $\frac{3}{8}\times 16 = 6$

Therefore, option (A) is correct.

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10. If $f(x)=\int_0^x t\sin t\ dt$, **then** $f'(x)$ **is:**

(A) $\cos x + x\sin x$

(B) $x\sin x$

(C) $x\cos x$

(D) $\sin x + x\cos x$

Ans. $f(x) = \int_0^x t\sin t \, dt$

= $\{t(-\cos t)\}_0^x - \int_0^x 1(-\cos t) \, dt$

[Applying Product Rule]

= $-x\cos x - 0 + \int_0^x \cos t \, dt$

= $-x\cos x + (\sin t)_0^x$

= $-x\cos x + \sin x - \sin 0°$

= $-x\cos x + \sin x$

$\therefore f'(x) = -\{x(-\sin x) + (\cos x)1\} + \cos x$

= $x\sin x - \cos x + \cos x = x\sin x$

Therefore, option (B) is correct.