CBSE Class–12 Mathematics

NCERT solution

Chapter - 7

Integrals - Miscellaneous Exercise

Integrate the function in Exercises 1 to 11.

1. $\frac{1}{x-x^3}$

Ans. Let I = $\int \frac{1}{x-x^3} dx$

Here, $\frac{1}{x-x^3}$

$= \frac{1}{x(1-x^2)}$

$= \frac{1}{x(1-x)(1+x)}$

$= \frac{A}{x}+\frac{B}{1-x}+\frac{C}{1+x}$ ....(i)

$\Rightarrow\ 1 = A(1-x)(1+x)+Bx(1+x)+Cx(1-x)$

$\Rightarrow\ 1 = A(1-x^2)+B(x+x^2)+C(x-x^2)$

$\Rightarrow\ 1 = A - Ax^2 + Bx + Bx^2 + Cx - Cx^2$

Comparing the coefficients of x^2 –A + B – C = 0 ....(ii)

Comparing the coefficients of x B + C = 0 ....(iii)

Comparing constants A = 1 ....(iv)

On solving eq. (ii), (iii) and (iv), we get A = 1, B = $\frac{1}{2}$, C = $\frac{-1}{2}$

Putting these values in eq. (i),

$$\frac{1}{x-x^3}$$

$$= \frac{1}{x}+\frac{\frac{1}{2}}{1-x}+\frac{\frac{-1}{2}}{1+x}$$

$$\Rightarrow \int\frac{1}{x-x^3}\,dx=\int\frac{1}{x}\,dx+\frac{1}{2}\int\frac{1}{1-x}\,dx-\frac{1}{2}\int\frac{1}{1+x}\,dx$$

$$= \log|x|+\frac{1}{2}\frac{\log|1-x|}{-1}-\frac{1}{2}\log|1+x|+c$$

$$= \frac{1}{2}\left[2\log|x|-\log|1-x|-\log|1+x|\right]+c$$

$$= \frac{1}{2}\left[\log|x|^2-\left(\log|1-x|+\log|1+x|\right)\right]+c$$

$$= \frac{1}{2}\left[\log|x|^2-\log|1-x||1+x|\right]+c$$

$$= \frac{1}{2}\left[\log|x|^2-\log|1-x^2|\right]+c$$

$$= \frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right|+c \text{ Ans.}$$

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2. $\frac{1}{\sqrt{x+a}+\sqrt{x+b}}$

Ans. Let I = $\int\frac{1}{\sqrt{x+a}+\sqrt{x+b}}\,dx$

$= \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}} \times \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}\ dx$

$= \int \frac{\sqrt{x+a}-\sqrt{x+b}}{x+a-x-b}\ dx$

$= \int \frac{\sqrt{x+a}-\sqrt{x+b}}{a-b}\ dx$

$= \frac{1}{a-b}\int\left(\sqrt{x+a}-\sqrt{x+b}\right)\ dx$

$= \frac{1}{a-b}\left[\int(x+a)^{\frac{1}{2}}\ dx-\int(x+b)^{\frac{1}{2}}\ dx\right]$

$= \frac{1}{a-b}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}(1)}-\frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}(1)}\right]+c$

$= \frac{1}{a-b}\left[\frac{2}{3}(x+a)^{\frac{3}{2}}-\frac{2}{3}(x+b)^{\frac{3}{2}}\right]+c$

$= \frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}}-(x+b)^{\frac{3}{2}}\right]+c$ Ans.

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3. $\frac{1}{x\sqrt{ax-x^2}}$

Ans. Let I = $\int \frac{1}{x\sqrt{ax-x^2}}\ dx$ ....(i)

Putting $x=\frac{1}{t}=t^{-1}$

$\Rightarrow \ dx = \frac{-1}{t^2}\ dt$

$\therefore$ From eq. (i),

I = $\int \frac{\frac{-1}{t^2}\ dt}{\frac{1}{t}\sqrt{\frac{a}{t} - \frac{1}{t^2}}}$

= $-\int \frac{dt}{\sqrt{at-1}}$

= $-\int (at-1)^{\frac{-1}{2}}\ dt$

= $\frac{-(at-1)^{\frac{1}{2}}}{\frac{1}{2}\times a} + c$

= $\frac{-2}{a}\sqrt{\frac{a}{x} - 1} + c$

= $\frac{-2}{a}\sqrt{\frac{a-x}{x}} + c$ Ans.

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4. $\frac{1}{x^2\left(x^4+1\right)^{3/4}}$

Ans. Let I = $\int \frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}\ dx$

= $\int \frac{1}{x^2\left[x^4\left(1+\frac{1}{x^4}\right)\right]^{\frac{3}{4}}}\ dx$

= $\int \frac{1}{x^2.x^3\left(1+\frac{1}{x^4}\right)^{\frac{3}{4}}}\ dx$

= $\int \frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{\frac{-3}{4}}\ dx$

Putting $1+\frac{1}{x^4}=t$

$\Rightarrow\ -4x^{-5}\ dx = dt$

$\Rightarrow\ \frac{1}{x^5}\ dx = \frac{-1}{4}\ dt$

$\therefore$ I = $\frac{-1}{4}\int t^{\frac{-3}{4}}\ dt$

= $\frac{-1}{4}.\frac{t^{1/4}}{1/4}+c$

= $-\left(1+\frac{1}{x^4}\right)^{1/4}+c$ Ans.

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5. $\frac{1}{x^{1/2}+x^{1/3}}$

Ans. Let I = $\int \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}\ dx$

Putting $x^{\frac{1}{6}} = t$

$\Rightarrow x = t^6$

$\Rightarrow dx = 6t^5 \ dt$

$\therefore$ I = $\int \frac{6t^5}{t^3 + t^2} \ dt$

= $6\int \frac{t^5}{t^2(t+1)} \ dt$

= $6\int \frac{t^3}{t+1} \ dt$

= $6\int \frac{(t^3+1)-1}{t+1} \ dt$

= $6\left[\int \frac{t^3+1}{t+1} - \frac{1}{t+1} \ dt \right]$

= $6\left[\int \left(\frac{(t+1)(t^2-t+1)}{t+1} - \frac{1}{t+1} \right) dt \right]$

= $6\left[\int \left(t^2 - t + 1 - \frac{1}{t+1} \right) dt \right]$

= $6\left[\frac{t^3}{3} - \frac{t^2}{2} + t - \log|t+1| \right] + c$

= $2t^3 - 3t^2 + 6t - 6\log|t+1| + c$

= $2\sqrt{x} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\log\left|x^{\frac{1}{6}} + 1\right| + c$ Ans.

6. $\dfrac{5x}{(x+1)(x^2+9)}$

Ans. Let I = $\displaystyle\int \frac{5x}{(x+1)(x^2+9)}\,dx$ ....(i)

Let $\dfrac{5x}{(x+1)(x^2+9)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+9}$ ....(ii)

$\Rightarrow\ 5x = A(x^2+9) + (Bx+C)(x+1)$

$\Rightarrow\ 5x = Ax^2 + 9A + Bx^2 + Bx + Cx + C$

Comparing coefficients of x^2 A + B = 0 ....(iii)

Comparing coefficients of x B + C = 5 ....(iv)

Comparing constants 9A + C = 0 ....(v)

On solving eq. (iii), (iv) and (v), we get A = $\dfrac{-1}{2}$, B = $\dfrac{1}{2}$, C = $\dfrac{9}{2}$

Putting these values of A, B and C in eq. (ii),

$$\frac{5x}{(x+1)(x^2+9)} = \frac{\frac{-1}{2}}{x+1} + \frac{\frac{1}{2}x+\frac{9}{2}}{x^2+9}$$

$\therefore$ From eq. (i),

I = $\displaystyle\int \frac{5x}{(x+1)(x^2+9)}\,dx = \frac{-1}{2}\int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{x}{x^2+9}\,dx + \frac{9}{2}\int \frac{1}{x^2+9}\,dx$

= $\displaystyle\frac{-1}{2}\log|x+1| + \frac{1}{4}\int \frac{2x}{x^2+9}\,dx + \frac{9}{2}\cdot\frac{1}{3}\tan^{-1}\frac{x}{3} + c$

$= \frac{-1}{2}\log|x+1| + \frac{1}{4}\log|x^2+9| + \frac{3}{2}\tan^{-1}\frac{x}{3} + c$

$= \frac{-1}{2}\log|x+1| + \frac{1}{4}\log(x^2+9) + \frac{3}{2}\tan^{-1}\frac{x}{3} + c$ Ans.

7. $\frac{\sin x}{\sin(x-a)}$

Ans. Let I = $\int \frac{\sin x}{\sin(x-a)}\, dx$

$= \int \frac{\sin(x-a+a)}{\sin(x-a)}\, dx$

$= \int \frac{\sin(x-a)\cos a + \cos(x-a)\sin a}{\sin(x-a)}\, dx$

$= \int \left(\frac{\sin(x-a)\cos a}{\sin(x-a)} + \frac{\cos(x-a)\sin a}{\sin(x-a)} \right) dx$

$= \int (\cos a + \sin a \cot(x-a))\, dx$

$= \int \cos a\, dx + \int \sin a \cot(x-a)\, dx$

$= \cos a \int 1\, dx + \sin a \int \cot(x-a)\, dx$

$= (\cos a)x + \sin a \frac{\log|\sin(x-a)|}{1} + c$

$= x\cos a + \sin a \log|\sin(x-a)| + c$ Ans.

8. $\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$

Ans. Let I = $\int \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}\ dx$

= $\int \frac{e^{\log x^5} - e^{\log x^4}}{e^{\log x^3} - e^{\log x^2}}\ dx$

= $\int \frac{x^5 - x^4}{x^3 - x^2}\ dx$

= $\int \frac{x^4(x-1)}{x^2(x-1)}\ dx$

$\Rightarrow$ I = $\int x^2\ dx$

= $\frac{x^3}{3} + c$ Ans.

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9. $\frac{\cos x}{\sqrt{4 - \sin^2 x}}$

Ans. Let I = $\int \frac{\cos x}{\sqrt{4 - \sin^2 x}}\ dx$ ....(i)

Putting $\sin x = t$

$\Rightarrow \cos x = \frac{dt}{dx}$

$\Rightarrow \cos x\, dx = dt$

$\therefore$ From eq. (i),

I = $\int \frac{dt}{\sqrt{4 - t^2}}$

= $\sin^{-1}\left(\frac{t}{2}\right)+c$

= $\sin^{-1}\left[\frac{1}{2}\sin x\right]+c$ Ans.

10. $\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}$

Ans. Let I = $\int \frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}\,dx$ ....(i)

$\Rightarrow$ I = $\int \frac{\left(\sin^4 x\right)^2-\left(\cos^4 x\right)^2}{1-2\sin^2 x\cos^2 x}\,dx$

= $\int \frac{\left(\sin^4 x-\cos^4 x\right)\left(\sin^4 x+\cos^4 x\right)}{1-2\sin^2 x\cos^2 x}\,dx$

$\Rightarrow$ I = $\int \frac{\left\{\left(\sin^2 x\right)^2-\left(\cos^2 x\right)^2\right\}\left\{\left(\sin^2 x\right)^2+\left(\cos^2 x\right)^2\right\}}{1-2\sin^2 x\cos^2 x}\,dx$

$\Rightarrow$ I = $\int \frac{\left(\sin^2 x-\cos^2 x\right)\left(\sin^2 x+\cos^2 x\right)\left\{\left(\sin^2 x+\cos^2 x\right)^2-2\sin^2 x\tan^2 x\right\}}{1-2\sin^2 x\cos^2 x}\,dx$

$\Rightarrow$ I = $\int \frac{-\left(\cos^2 x-\sin^2 x\right)\left\{1-2\sin^2 x\tan^2 x\right\}}{1-2\sin^2 x\cos^2 x}\,dx$

= $\int \frac{-\cos 2x\left\{1-2\sin^2 x\tan^2 x\right\}}{1-2\sin^2 x\cos^2 x}\,dx$

$\Rightarrow$ I = $-\int \cos 2x\,dx$

= $\frac{-\sin 2x}{2}+c$ Ans.

11. $\frac{1}{\cos(x+a)\cos(x+b)}$

Ans. Let I = $\int \frac{1}{\cos(x+a)\cos(x+b)}\,dx$ ....(i)

$\Rightarrow$ I = $\frac{1}{\sin(a-b)}\int \frac{\sin(a-b)}{\cos(x+a)\cos(x+b)}\,dx$

= $\frac{1}{\sin(a-b)}\int \frac{\sin\left[(x+a)-(x+b)\right]}{\cos(x+a)\cos(x+b)}\,dx$

$\Rightarrow$ I = $\frac{1}{\sin(a-b)}\int \frac{\sin(x+a)\cos(x+b)-\cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}\,dx$

$\Rightarrow$ I = $\frac{1}{\sin(a-b)}\int \left(\frac{\sin(x+a)\cos(x+b)}{\cos(x+a)\cos(x+b)}-\frac{\cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}\right)dx$

$\Rightarrow$ I = $\frac{1}{\sin(a-b)}\int \left(\tan(x+a)-\tan(x+b)\right)dx$

$\Rightarrow$ I = $\frac{1}{\sin(a-b)}\left[-\log\left|\cos(x+a)\right|+\log\left|\cos(x+b)\right|\right]+c$

$\Rightarrow$ I = $\frac{1}{\sin(a-b)}\log\left|\frac{\cos(x+b)}{\cos(x+a)}\right|+c$ Ans.

Integrate the function in Exercises 12 to 22.

12. $\frac{x^3}{\sqrt{1-x^8}}$

Ans. Let I = $\int \frac{x^3}{\sqrt{1-x^8}}\ dx$

= $\frac{1}{4}\int \frac{4x^3}{\sqrt{1-x^8}}\ dx$ ....(i)

Putting $x^4 = t$

$\Rightarrow\ 4x^3 = \frac{dt}{dx}$

$\Rightarrow\ 4x^3\ dx = dt$

$\therefore$ From eq. (i),

I = $\frac{1}{4}\int \frac{dt}{\sqrt{1-t^2}}$

= $\frac{1}{4}\sin^{-1} t + c$

= $\frac{1}{4}\sin^{-1}\left(x^4\right) + c$ Ans.

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13. $\frac{e^x}{\left(1+e^x\right)\left(2+e^x\right)}$

Ans. Let I = $\int \frac{e^x}{\left(1+e^x\right)\left(2+e^x\right)}\ dx$ ....(i)

Putting $e^x = t$

$\Rightarrow\ e^x = \frac{dt}{dx}$

$\Rightarrow\ e^x\ dx = dt$

$\therefore$ From eq. (i), I = $\int \frac{dt}{(1+t)(2+t)}$

= $\int \frac{1}{(t+1)(t+2)}\, dt$ ....(ii)

$=> I = \int \frac{(t+2)-(t+1)}{(t+1)(t+2)} dt$

$=> I = \int \left(\frac{t+2}{(t+1)(t+2)} - \frac{t+1}{(t+1)(t+2)} \right) dt$

$\Rightarrow$ I = $\int \left(\frac{1}{(t+1)} - \frac{1}{(t+2)} \right) dt$

= $\log|t+1| - \log|t+2| + c$

$\Rightarrow$ I = $\log\left|\frac{t+1}{t+2}\right| + c$

= $\log\left|\frac{e^x+1}{e^x+2}\right| + c$ Ans.

14. $\frac{1}{(x^2+1)(x^2+4)}$

Ans. Let I = $\int \frac{1}{(x^2+1)(x^2+4)}\, dx$ ....(i)

$\Rightarrow$ I = $\frac{1}{3}\int \frac{3}{(x^2+1)(x^2+4)}\, dx$

= $\frac{1}{3}\int \frac{(x^2+4)-(x^2+1)}{(x^2+1)(x^2+4)}\, dx$

$\Rightarrow$ I = $\frac{1}{3}\int\left(\frac{1}{\left(x^2+1\right)}-\frac{1}{\left(x^2+4\right)}\right)dx$

= $\frac{1}{3}\left[\int\frac{1}{x^2+1}\ dx-\int\frac{1}{x^2+4}\ dx\right]$

$\Rightarrow$ I = $\frac{1}{3}\left[\tan^{-1}x-\frac{1}{2}\tan^{-1}\frac{x}{2}\right]$ +c Ans.

15. $\cos^3 xe^{\log\sin x}$

Ans. Let I = $\int\cos^3 x\ e^{\log\sin x}\ dx$

= $\int\cos^3 x\sin x\ dx$

= $-\int\cos^3 x\left(-\sin x\right)\ dx$ ....(i)

Putting $\cos x = t$

$\Rightarrow\ -\sin x=\frac{dt}{dx}$

$\Rightarrow\ -\sin x\ dx = dt$

$\therefore$ From eq. (i),

I = $-\int t^3\ dt$

= $\frac{-t^4}{4}+c$

= $\frac{-1}{4}\cos^4 x+c$ Ans.

16. $e^{3\log x}\left(x^4+1\right)^{-1}$

Ans. Let I = $\int e^{3\log x}\left(x^4+1\right)^{-1}\ dx$

= $\int \frac{e^{3\log x}}{\left(x^4+1\right)}\ dx$

= $\int \frac{e^{\log x^3}}{\left(x^4+1\right)}\ dx$

= $\int \frac{x^3}{\left(x^4+1\right)}\ dx$

$\Rightarrow$ I = $\frac{1}{4}\int \frac{4x^3}{\left(x^4+1\right)}\ dx$

Putting $x^4+1=t$

$\Rightarrow 4x^3=\frac{dt}{dx}$

$\Rightarrow 4x^3\ dx=dt$

$\therefore$ From eq. (i),

I = $\frac{1}{4}\int \frac{dt}{t}$

= $\frac{1}{4}\log|t|+c$

$\Rightarrow$ I = $\frac{1}{4}\log\left|x^4+1\right|+c$

= $\frac{1}{4}\log\left(x^4+1\right)+c$ Ans.

17. $\int f'(ax+b)\left(f(ax+b)\right)^n \, dx$

Ans. Let I = $\int f'(ax+b)\left\{f(ax+b)\right\}^n \, dx$

= $\frac{1}{a}\int f'(ax+b)^n \, af(ax+b) \, dx$ ....(i)

Putting $f(ax+b)=t$

$\Rightarrow f'(ax+b)\frac{d}{dx}(ax+b)=\frac{dt}{dx}$

$\Rightarrow af'(ax+b) \, dx = dt$

$\therefore$ From eq. (i),

I = $\frac{1}{a}\int t^n \, dt$

= $\frac{1}{a}.\frac{t^{n+1}}{n+1}+c$ if $n\neq -1$

$\Rightarrow$ I = $\frac{\left\{f(ax+b)\right\}^{n+1}}{a(n+1)}+c$ Ans.

18. $\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$

Ans. Let I = $\int \frac{dx}{\sqrt{\sin^3 x \sin(x+\alpha)}}$

= $\int \frac{dx}{\sqrt{\sin^3 x(\sin x\cos\alpha+\cos x\sin\alpha)}}$

$$\Rightarrow \text{ I} = \int \frac{dx}{\sqrt{\sin^3 x.\sin x(\cos\alpha + \cot x \sin\alpha)}}$$

$$= \int \frac{dx}{\sin^2 x\sqrt{\cos\alpha + \cot x\sin\alpha}}$$

$$\Rightarrow \text{ I} = \int \frac{\cos ec^2 x\ dx}{\sqrt{\cos\alpha + \cot x\sin\alpha}}$$

Putting $\cos\alpha + \cot x\sin\alpha = t$

$$\Rightarrow -\cos ec^2 x\sin\alpha\ dx = dt$$

$$\Rightarrow \cos ec^2 x\ dx = -\frac{dt}{\sin\alpha}$$

$$\therefore \text{ I} = -\int \frac{dt}{\sin\alpha\sqrt{t}}$$

$$= \frac{-1}{\sin\alpha}\int t^{\frac{-1}{2}}\ dt$$

$$= \frac{-1}{\sin\alpha}.\frac{t^{1/2}}{1/2} + c$$

$$\Rightarrow \text{ I} = \frac{-2}{\sin\alpha}.\sqrt{\cos\alpha + \cot x\sin\alpha} + c$$

$$= \frac{-2}{\sin\alpha}.\sqrt{\cos\alpha + \frac{\cos x}{\sin x}\sin\alpha} + c$$

$$\Rightarrow \text{ I} = \frac{-2}{\sin\alpha}.\sqrt{\frac{\sin x\cos\alpha + \cos x\sin\alpha}{\sin x}} + c$$

$$= \frac{-2}{\sin\alpha}.\sqrt{\frac{\sin(x+\alpha)}{\sin x}} + c \text{ Ans.}$$

19. $\dfrac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}, x\in[0,1]$

Ans. We know that $\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}=\dfrac{\pi}{2}$

$\Rightarrow \cos^{-1}\sqrt{x}=\dfrac{\pi}{2}-\sin^{-1}\sqrt{x}$

$=> I=\int \dfrac{sin^{-1}\sqrt{x}-(\frac{\pi}{2}-sin^{-1}\sqrt{x})}{\frac{\pi}{2}}dx$

$= \dfrac{2}{\pi}\int\left(2\sin^{-1}\sqrt{x}-\dfrac{\pi}{2}\right)dx$

$\Rightarrow$ I = $\dfrac{4}{\pi}\int \sin^{-1}\sqrt{x}\ dx-\int 1\ dx$

$= \dfrac{4}{\pi}\int \sin^{-1}\sqrt{x}\ dx - x + c$ ....(i)

Putting $\sqrt{x}=\sin\theta$

$\Rightarrow x=\sin^2\theta$

$\Rightarrow dx=2\sin\theta\cos\theta\ d\theta = \sin 2\theta\ d\theta$

$\therefore$ I = $\dfrac{4}{\pi}\int\left(\sin^{-1}(\sin\theta).\sin 2\theta\right)d\theta - x + c$

$= \dfrac{4}{\pi}\int(\theta.\sin 2\theta)\ d\theta - x + c$

$\Rightarrow$ I = $\dfrac{4}{\pi}\int\left[\theta\left(\dfrac{-\cos 2\theta}{2}\right)-\int 1.\left(\dfrac{-\cos 2\theta}{2}\right)d\theta\right]-x+c$

[Applying product rule]

$$\Rightarrow \text{I} = \frac{4}{\pi}\left[-\frac{1}{2}\theta\cos 2\theta + \frac{1}{2}\int\cos 2\theta\ d\theta\right] - x + c$$

$$\Rightarrow \text{I} = \frac{4}{\pi}\left[-\frac{1}{2}\theta\left(1-2\sin^2\theta\right) + \frac{1}{4}2\sin\theta\cos\theta\right] - x + c$$

$$\Rightarrow \text{I} = \frac{4}{\pi}\left[-\frac{1}{2}\theta\left(1-2\sin^2\theta\right) + \frac{1}{2}\sin\theta\sqrt{1-\sin^2\theta}\right] - x + c$$

Putting $\sin\theta = \sqrt{x}$

$$\Rightarrow \text{I} = \frac{4}{\pi}\left[-\frac{1}{2}\left(\sin^{-1}\sqrt{x}\right)(1-2x) + \frac{1}{2}\sqrt{x}\sqrt{1-x}\right] - x + c$$

$$\Rightarrow \text{I} = \frac{4}{\pi}\left[-\frac{1}{2}(1-2x)\sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x}\sqrt{1-x}\right] - x + c$$

$$\Rightarrow \text{I} = -\frac{2}{\pi}(1-2x)\sin^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x}\sqrt{1-x} - x + c$$

$$= \frac{2(2x-1)}{\pi}sin^{-1}\sqrt{x} + \frac{2\sqrt{x-x^2}}{\pi} - x + c\ Ans.$$

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20. $\sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}$

Ans. Let I = $\int\sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}\ dx$

Putting $\sqrt{x} = t$

$\Rightarrow x = t^2$

$\Rightarrow dx = 2t\ dt$

$\therefore$ I = $\int\sqrt{\dfrac{1-t}{1+t}}\,2t\ dt$

$= 2\int t\sqrt{\frac{1-t}{1+t}}\ dt$

$= 2\int t\sqrt{\frac{1-t}{1+t}\times\frac{1-t}{1-t}}\ dt$

$= 2\int \frac{t(1-t)}{\sqrt{1-t^2}}\ dt$

$\Rightarrow$ I = $2\int \frac{t-t^2}{\sqrt{1-t^2}}\ dt$ ....(i)

$\Rightarrow$ I = $2\int \frac{(1-t^2)+t-1}{\sqrt{1-t^2}}\ dt$

$= 2\left[\int\sqrt{1-t^2}\ dt+\int\frac{t}{\sqrt{1-t^2}}\ dt-\int\frac{1}{\sqrt{1-t^2}}\ dt\right]$

$\Rightarrow$ I = $2\left[\frac{t}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1}t+\int\frac{t}{\sqrt{1-t^2}}\ dt-\sin^{-1}t\right]+c$

$\Rightarrow$ I = $2\left[\frac{1}{2}t\sqrt{1-t^2}-\frac{1}{2}\sin^{-1}t+\int\frac{t}{\sqrt{1-t^2}}\ dt\right]+c$ ....(ii)

For evaluating $\int\frac{t}{\sqrt{1-t^2}}\ dt$, putting $1-t^2=z$

$\Rightarrow -2t\ dt = dz$

$\Rightarrow t\ dt = -\frac{1}{2}\ dz$

$\therefore \int\frac{t}{\sqrt{1-t^2}}\ dt$

$= \int \frac{\frac{-1}{2}}{\sqrt{z}} dz$

$= -\frac{1}{2} \int \frac{1}{\sqrt{z}} dz$

$= -\frac{1}{2} \int z^{-\frac{1}{2}} dz$

$= -\frac{1}{2} \cdot \frac{z^{1/2}}{1/2}$

$= -\sqrt{1-t^2}$ ....(iii)

Putting this value in eq. (ii),

$I = 2\left[\frac{1}{2} t\sqrt{1-t^2} - \frac{1}{2}\sin^{-1} t - \sqrt{1-t^2}\right] + c$

$=> I = t\sqrt{1-t^2} - sin^{-1}t - 2\sqrt{1-t^2} + c$

$= (t-2)\sqrt{1-t^2} - \sin^{-1} t + c$

$\Rightarrow I = \left(\sqrt{x} - 2\right)\sqrt{1-x} - \sin^{-1}\sqrt{x} + c$ Ans.

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21. $\frac{2+\sin 2x}{1+\cos 2x} e^x$

Ans. Let $I = \int \frac{2+\sin 2x}{1+\cos 2x} e^x \, dx$

$= \int e^x \frac{2+2\sin x \cos x}{2\cos^2 x} \, dx$

$= \int e^x \left(\frac{2}{2\cos^2 x} + \frac{2\sin x \cos x}{2\cos^2 x}\right) dx$

= $\int e^x\left(\frac{1}{\cos^2 x}+\frac{\sin x}{\cos x}\right) dx$

= $\int e^x\left(\sec^2 x+\tan x\right) dx$

= $\int e^x\left(\tan x+\sec^2 x\right) dx$

$\left(\int e^x\left(f(x)+f'(x)\right)\right)$

= $e^x \tan x+c$

22. $\frac{x^2+x+1}{(x+1)^2(x+2)}$

Ans. Let I = $\int \frac{x^2+x+1}{(x+1)^2(x+2)} dx$ ....(i)

Let $\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x+2}$ ....(ii)

$\Rightarrow\ x^2+x+1=A(x+1)(x+2)+B(x+2)+C(x+1)^2$

$\Rightarrow\ x^2+x+1=A\left(x^2+3x+2\right)+B(x+2)+C\left(x^2+1+2x\right)$

$\Rightarrow\ x^2+x+1=Ax^2+3Ax+2A+Bx+2B+Cx^2+C+2Cx$

Comparing coefficients of x^2 A + C = 1 ....(iii)

Comparing coefficients of x 3A + B + 2C = 1 ....(iv)

Comparing constants 2A + 2B + C = 1 ....(v)

On solving eq. (iii), (iv) and (v), we get A = –2, B = 1, C = 3

Putting these values of A, B and C in eq. (ii),

$$\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{-2}{x+1}+\frac{1}{(x+1)^2}+\frac{3}{x+2}$$

$$\therefore \int\frac{x^2+x+1}{(x+1)^2(x+2)}\,dx$$

$$= \int\left(\frac{-2}{x+1}+\frac{1}{(x+1)^2}+\frac{3}{x+2}\right)dx$$

$$= -\int\frac{1}{x+1}\,dx+\int(x+1)^{-2}\,dx+3\int\frac{1}{x+2}\,dx$$

$$= -2\log|x+1|+\frac{(x+1)^{-2+1}}{-2+1}+3\log|x+2|+c$$

$$= -2\log|x+1|-\frac{1}{x+1}+3\log|x+2|+c \text{ Ans.}$$

Evaluate the integrals in Exercises 23 and 24.

23. $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$

Ans. Let I = $\int\tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$ ....(i)

Putting $x=\cos 2\theta$

$$\Rightarrow \frac{dx}{d\theta}=-2\sin 2\theta$$

$$\Rightarrow dx=-2\sin 2\theta\, d\theta$$

And $\tan^{-1}\sqrt{\frac{1-x}{1+x}}=\tan^{-1}\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}$

$= \tan^{-1}\sqrt{\dfrac{2\sin^2\theta}{2\cos^2\theta}}$

$= \tan^{-1}\sqrt{\tan^2\theta}$

$= \tan^{-1}(\tan\theta) = \theta$

$\therefore$ I = $\int \theta(-2\sin 2\theta\, d\theta)$

$= -2\int \theta \sin 2\theta\, d\theta$

[Applying Product Rule]

$= -2\left[\theta\left(\dfrac{-\cos 2\theta}{2}\right) - \int 1\left(\dfrac{-\cos 2\theta}{2}\right) d\theta\right]$

$= -2\left[\dfrac{-1}{2}\theta\cos 2\theta + \dfrac{1}{2}\int \cos 2\theta\, d\theta\right]$

$= \theta\cos 2\theta - \dfrac{\sin 2\theta}{2} + c$

$= \theta\cos 2\theta - \dfrac{1}{2}\sqrt{1-\cos^2 2\theta} + c$

$= \theta\left(\cos^{-1}x\right)x - \dfrac{1}{2}\sqrt{1-x^2} + c$

$= \dfrac{1}{2}\left[x\cos^{-1}x - \sqrt{1-x^2}\right] + c$ Ans.

24. $\dfrac{\sqrt{x^2+1}\left[\log\left(x^2+1\right) - 2\log x\right]}{x^4}$

Ans. Let I = $\int \frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right)-2\log x\right]}{x^4}\ dx$

= $\int \frac{\sqrt{x^2+1}}{x^4}\left[\log\left(x^2+1\right)-\log x^2\right]\ dx$

= $\int \frac{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{x^4}\log\left(\frac{x^2+1}{x^2}\right)\ dx$

= $\int \frac{\sqrt{\left(1+\frac{1}{x^2}\right)}}{x^3}\log\left(1+\frac{1}{x^2}\right)\ dx$

= $\int \sqrt{\left(1+\frac{1}{x^2}\right)}\log\left(1+\frac{1}{x^2}\right)\ \frac{dx}{x^3}$

Putting $1+\frac{1}{x^2}=t$

$\Rightarrow\ 1+x^{-2}=t$

$\Rightarrow\ \frac{-2}{x^3}\ dx=dt$

$\Rightarrow\ \frac{dx}{x^3}=-\frac{1}{2}\ dt$

$\therefore$ I = $-\frac{1}{2}\int \sqrt{t}\log t\ dt$

= $-\frac{1}{2}\int (\log t).t^{\frac{1}{2}}\ dt$

[Applying Product Rule]

$$= -\frac{1}{2}\left[(\log t).\frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \int \frac{1}{t}.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}dt\right]$$

$$\Rightarrow \text{I} = -\frac{1}{3}t^{3/2}\log t + \frac{1}{3}\int t^{1/2}\,dt$$

$$= -\frac{1}{3}t^{3/2}\log t + \frac{1}{3}.\frac{t^{3/2}}{3/2} + c$$

$$\Rightarrow \text{I} = \frac{2}{9}t^{3/2} - \frac{1}{3}t^{3/2}\log t + c$$

$$= \frac{1}{3}t^{3/2}\left[\frac{2}{3} - \log t\right] + c$$

$$= \frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\left[\frac{2}{3} - \log\left(1+\frac{1}{x^2}\right)\right] + c \text{ Ans.}$$

Evaluate the definite integrals in Exercise 25 to 33.

25. $\int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$

Ans. Let I = $\int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$

$$= \int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx$$

= $\int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1}{2\sin^2 \frac{x}{2}} - \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}} \right) dx$

= $\int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1}{2}\operatorname{cosec}^2 \frac{x}{2} - \cot\frac{x}{2} \right) dx$

= $\int_{\frac{\pi}{2}}^{\pi} e^x \left(-\cot\frac{x}{2} + \frac{1}{2}\operatorname{cosec}^2 \frac{x}{2} \right) dx$

$\left[\because \int e^x \left(f(x) + f'(x) \right) dx = e^x f(x) \right]$

= $\left(-e^x \cot\frac{x}{2} \right)_{\frac{\pi}{2}}^{\pi}$

= $-e^\pi \cot\frac{\pi}{2} - \left(-e^{\frac{\pi}{2}} \cot\frac{\pi}{4} \right)$

= $e^{\frac{\pi}{2}}$ Ans.

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26. $\int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$

Ans. Let I = $\int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$

divide numerator and denominator by $\cos^4 x$

$\Rightarrow$ I = $\int_0^{\frac{\pi}{4}} \frac{\frac{\sin x \cos x}{\cos x \cos x \cos^2 x}}{1+\frac{\sin^4 x}{\cos^4 x}} dx$

$\Rightarrow$ I = $\int_0^{\frac{\pi}{4}} \frac{\tan x \sec^2 x}{1+\tan^4 x} dx$

= $\frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{2\tan x \sec^2 x}{1+\tan^4 x} dx$ ....(i)

Putting $\tan^2 x = t$

$\Rightarrow 2\tan x \frac{d}{dx}(\tan x) = \frac{dt}{dx}$

$\Rightarrow 2\tan x \sec^2 x\, dx = dt$

Limits of integration when $x=0, t=\tan^2 x = \tan^2 0° = 0$ and when $x=\frac{\pi}{4}, t=\tan^2\frac{\pi}{4}=1$

$\therefore$ I = $\frac{1}{2}\int_0^1 \frac{dt}{1+t^2}$

= $\frac{1}{2}\left(\tan^{-1} t\right)_0^1$

= $\frac{1}{2}\left(\tan^{-1} 1 - \tan^{-1} 0\right)$

= $\frac{1}{2}\left(\frac{\pi}{4}-0\right) = \frac{\pi}{8}$ Ans.

27. $\int_0^{\frac{\pi}{2}} \frac{\cos^2 x\, dx}{\cos^2 x + 4\sin^2 x}$

Ans. Let I = $\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x}\, dx$

[Dividing each term by $\cos^2 x$]

= $\int_0^{\frac{\pi}{2}} \frac{1}{1+4\tan^2 x}\, dx$ ....(i)

Putting $\tan x = t$

$\Rightarrow \sec^2 x = \frac{dt}{dx}$

$\Rightarrow dx = \frac{dt}{\sec^2 x} = \frac{dt}{1+\tan^2 x}$

= $\frac{dt}{1+t^2}$

Limits of integration when $x=0, t=\tan 0^\circ = 0$ and when $x=\frac{\pi}{2}, t=\tan\frac{\pi}{2}=\infty$

$\therefore$ I = $\int_0^{\infty} \frac{1}{1+4t^2} \cdot \frac{dt}{1+t^2}$

= $\int_0^{\infty} \frac{1}{(1+4t^2)(1+t^2)}\, dt$ ....(ii)

$\Rightarrow$ I = $\frac{1}{3}\int_0^{\infty} \frac{3}{(1+4t^2)(1+t^2)}\, dt$

= $\frac{1}{3}\int_0^\infty \frac{4\left(t^2+1\right)-\left(4t^2+1\right)}{\left(1+4t^2\right)\left(1+t^2\right)}\ dt$

$\Rightarrow$ I = $\frac{1}{3}\int_0^\infty \left[\frac{4\left(t^2+1\right)}{\left(1+4t^2\right)\left(1+t^2\right)}-\frac{\left(4t^2+1\right)}{\left(1+4t^2\right)\left(1+t^2\right)}\right] dt$

= $\frac{1}{3}\left[\int_0^\infty 4\frac{1}{\left(4t^2+1\right)}\ dt-\int_0^\infty \frac{1}{\left(1+t^2\right)}\ dt\right]$

$\Rightarrow$ I = $\frac{1}{3}\left[\int_0^\infty 4\frac{1}{\left(\left(2t\right)^2+1\right)}\ dt-\tan^{-1}t\right]$

$=\frac{1}{3}\left[4.\frac{\tan^{-1}2t}{2}-\tan^{-1}t\right]_0^\infty$

$=\frac{1}{3}[2(tan^{-1}\infty-tan^{-1}0)-(tan^{-1}\infty-tan^{-1}0)]$

$=\frac{1}{3}[2(\frac{\pi}{2}-0)-(\frac{\pi}{2}-0)]$

$=\frac{1}{3}\times\frac{\pi}{2}$

= $=\frac{\pi}{6}\ Ans.$

---

28. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x+\cos x}{\sqrt{\sin 2x}}\ dx$

Ans. Let I = $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x+\cos x}{\sqrt{\sin 2x}}\ dx$..........(i)

Putting $\sin x-\cos x=t$

$\Rightarrow\ \left(\cos x+\sin x\right)\ dx=dt$

Again $\left(\sin x-\cos x\right)^{2}=t^{2}$

$\Rightarrow \sin^{2}x+\cos^{2}x-2\sin x\cos x=t^{2}$

$\Rightarrow \sin 2x=1-t^{2}$

Limits of integration when $x=\frac{\pi}{6}, t=\sin\frac{\pi}{6}-\cos\frac{\pi}{6}=\frac{1}{2}-\frac{\sqrt{3}}{2}=\frac{-\left(\sqrt{3}-1\right)}{2}=-\alpha$ (say)

where $\alpha=\frac{\sqrt{3}-1}{2}$ ....(ii)

when $x=\frac{\pi}{3}, t=\sin\frac{\pi}{3}-\cos\frac{\pi}{3}=\frac{\sqrt{3}}{2}-\frac{1}{2}=\frac{\left(\sqrt{3}-1\right)}{2}=\alpha$

$\therefore$ I = $\int_{-\alpha}^{\alpha}\frac{dt}{\sqrt{1-t^{2}}}$ = $\left[\sin^{-1}t\right]_{-\alpha}^{\alpha}$

= $\sin^{-1}\alpha+\sin^{-1}\alpha$ = $2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$ [From eq. (ii) Ans.

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29. $\int_{0}^{1}\frac{dx}{\sqrt{1+x}-\sqrt{x}}$

Ans. Let I = $\int_{0}^{1}\frac{dx}{\sqrt{1+x}-\sqrt{x}}$

= $\int_{0}^{1}\frac{\sqrt{1+x}+\sqrt{x}}{\left(\sqrt{1+x}+\sqrt{x}\right)\left(\sqrt{1+x}-\sqrt{x}\right)}\,dx$

= $\int_{0}^{1}\frac{\sqrt{1+x}+\sqrt{x}}{1+x-x}\,dx$

$\Rightarrow$ I = $\int_0^1 \left(\sqrt{1+x}+\sqrt{x}\right)\ dx$

= $\int_0^1 (1+x)^{\frac{1}{2}}\ dx + \int_0^1 (x)^{\frac{1}{2}}\ dx$

= $\dfrac{\left[(1+x)^{\frac{3}{2}}\right]_0^1}{\frac{3}{2}(1)} + \dfrac{\left[(x)^{\frac{3}{2}}\right]_0^1}{\frac{3}{2}(1)}$

$\Rightarrow$ I = $\frac{2}{3}\left[(2)^{\frac{3}{2}}-(1)^{\frac{3}{2}}\right]+\frac{2}{3}\left[(1)^{\frac{3}{2}}-0\right]$

= $\frac{2}{3}\left[2\sqrt{2}-1\right]+\frac{2}{3}[1-0]$

$\Rightarrow$ I = $\frac{4\sqrt{2}}{3}-\frac{2}{3}+\frac{2}{3}=\frac{4\sqrt{2}}{3}$ Ans.

30. $\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{9+16\sin 2x}\ dx$

Ans. Let I = $\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{9+16\sin 2x}\ dx$

Putting $\sin x-\cos x=t$

$\Rightarrow (\cos x+\sin x)\ dx=dt$

Again $(\sin x-\cos x)^2=t^2$

$\Rightarrow \sin^2 x+\cos^2 x-2\sin x\cos x=t^2$

$\Rightarrow \sin 2x = 1 - t^2$

Limits of integration when $x = 0, t = 0 - 1 = -1$ and

when $x = \frac{\pi}{4}, t = \sin\frac{\pi}{4} - \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$

$\therefore$ I = $\int_{-1}^{0} \frac{dt}{9+16\left(1-t^2\right)}$

= $\int_{-1}^{0} \frac{dt}{25-16t^2}$

= $\int_{-1}^{0} \frac{dt}{16\left(\frac{25}{16}-t^2\right)}$

= $\frac{1}{16}\int_{-1}^{0} \frac{dt}{\left(\frac{5}{4}\right)^2 - t^2}$

$\Rightarrow$ I = $\frac{1}{16} \times \left[\frac{1}{2\times\frac{5}{4}} \log\left|\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right|\right]_{-1}^{0}$

= $\frac{1}{40}\left[\log 1 - \log\frac{1/4}{9/4}\right] = \frac{1}{40}\left[0 - \log\frac{1}{9}\right]$

$\Rightarrow$ I = $\frac{1}{40}\left[-\left(\log 1 - \log 9\right)\right] = \frac{1}{40}\log 9$ Ans.

31. $\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\ dx$

Ans. Let I = $\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\ dx$

= $\int_0^{\frac{\pi}{2}} 2\sin x \cos x \tan^{-1}(\sin x)\ dx$

Putting $\sin x = t$

$\Rightarrow\ \cos x\ dx = dt$

Limits of integration when $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = \sin\frac{\pi}{2} = 1$

$\therefore$ I = $2\int_0^1 t\tan^{-1} t\ dt$

= $2\int_0^1 \left(\tan^{-1} t\right) t\ dt$

[Applying Product Rule]

$\Rightarrow$ I = $2\left[\tan^{-1} t.\frac{t^2}{2} - \int \frac{1}{1+t^2}.\frac{t^2}{2}\ dt\right]$

$\Rightarrow$ I = $2\left[\frac{t^2}{2}\tan^{-1} t - \frac{1}{2}\int \frac{\left(1+t^2\right)-1}{1+t^2}\ dt\right]$

= $2\left[\frac{t^2}{2}\tan^{-1} t - \frac{1}{2}\int\left(1 - \frac{1}{1+t^2}\right)\ dt\right]$

$\Rightarrow$ I = $2\left[\frac{t^2}{2}\tan^{-1}t-\frac{1}{2}\left(t-\tan^{-1}t\right)\right]$

$= 2\left[\frac{t^2}{2}tan^{-1}t - \frac{1}{2}t + \frac{1}{2}tan^{-1}t\right]_0^1$

$= \left[(t^2+1)tan^{-1}t - t\right]_0^1$

= $\left(2\tan^{-1}1-1\right)-\left(0-0\right)$

= $2\times\frac{\pi}{4}-1$

= $\frac{\pi}{2}-1$ Ans.

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32. $\int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}\,dx$

Ans. Let I = $\int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}\,dx$

= $\int_0^{\pi}\frac{x\frac{\sin x}{\cos x}}{\frac{1}{\cos x}+\frac{\sin x}{\cos x}}\,dx$

= $\int_0^{\pi}\frac{x\sin x}{1+\sin x}\,dx$ ....(i)

$\Rightarrow$ I = $\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\sin(\pi-x)}\,dx$

= $\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\sin x}\,dx$ ....(ii)

Adding eq. (i) and (ii),

2I = $\int_0^{\pi} \frac{x\sin x+(\pi-x)\sin x}{1+\sin x}\,dx$

= $\int_0^{\pi} \frac{x\sin x+\pi\sin x-x\sin x}{1+\sin x}\,dx$

= $\int_0^{\pi} \frac{\pi\sin x}{1+\sin x}\,dx$

= $\pi\int_0^{\pi} \frac{\sin x}{1+\sin x}\,dx$

= $\pi\int_0^{\pi} \frac{(1+\sin x)-1}{1+\sin x}\,dx$

= $\pi\int_0^{\pi} \left(1-\frac{1}{1+\sin x}\right)dx$

$= \pi\int_0^{\pi} 1dx-\pi\int_0^{\pi} \frac{1}{1+\sin x}dx$

= $\pi(x)_0^{\pi}-2\pi\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$

= $\pi(\pi)-2\pi\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin\left(\frac{\pi}{2}-x\right)}$

= $\pi^2-2\pi\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x}$

= $\pi^2 - 2\pi\int_0^{\frac{\pi}{2}} \frac{dx}{2\cos^2\frac{x}{2}}$

= $\pi^2 - \pi\int_0^{\frac{\pi}{2}} \sec^2\frac{x}{2}\ dx$

= $\pi^2 - \pi\left[\frac{\tan\frac{x}{2}}{\frac{1}{2}}\right]_0^{\frac{\pi}{2}}$

=> 2I= $\pi^2 - 2\pi(1)$ = $\pi(\pi - 2)$

$\Rightarrow$ I = $\frac{\pi}{2}(\pi - 2)\ Ans.$

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33. $\int_1^4 \left[|x-1|+|x-2|+|x-3|\right]\ dx$

Ans. Let I = $\int_1^4 \left(|x-1|+|x-2|+|x-3|\right)\ dx$ ....(i)

If $x-1=0, x-2=0, x-3=0$ we get $x=1, x=2, x=3$

$\Rightarrow x = 2, 3 \in (1, 4)$

I =
$\int_1^2 \left(|x-1|+|x-2|+|x-3|\right)\ dx + \int_2^3 \left(|x-1|+|x-2|+|x-3|\right)\ dx + \int_3^4 \left(|x-1|+|x-2|+|x-3|\right)\ dx$

=
$\int_1^2 \{(x-1)-(x-2)-(x-3)\}\ dx + \int_2^3 \{(x-1)+(x-2)-(x-3)\}\ dx + \int_3^4 \{(x-1)+(x-2)+(x-3)\}\ dx$

$\Rightarrow$ I =

$$\int_1^2 (x-1-x+2-x+3)\ dx + \int_2^3 (x-1+x-2-x+3)\ dx + \int_3^4 (x-1+x-2+x-3)\ dx$$

$\Rightarrow$ I = $\int_1^2 (4-x)\ dx + \int_2^3 (x)\ dx + \int_3^4 (3x-6)\ dx$

= $\left(4x-\frac{x^2}{2}\right)_1^2 + \left(\frac{x^2}{2}\right)_2^3 + \left(\frac{3x^2}{2}-6x\right)_3^4$

$= (8-2) - (4-\frac{1}{2}) + \frac{9}{2} - \frac{4}{2} + (24-24) - (\frac{27}{2}-18)$

$= 6 - \frac{7}{2} + \frac{9}{2} - 2 + 0 + \frac{9}{2}$

$= 4 + \frac{11}{2}$

= $\frac{19}{2}$ Ans.

Prove the following (Exercise 34 to 40).

34. $\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log\frac{2}{3}$

Ans. Taking RHS

Let I = $\int_1^3 \frac{dx}{x^2(x+1)}$

= $\int_1^3 \frac{1}{x^2(x+1)}\ dx$ ....(i)

Let $\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$ ....(ii)

$\Rightarrow 1 = A(x)(x+1) + B(x+1) + C(x^2)$

$\Rightarrow 1 = A(x^2 + x) + Bx + B + Cx^2$

$\Rightarrow 1 = Ax^2 + Ax + Bx + B + Cx^2$

Comparing coefficients of x^2 A + C + 0 ....(iii)

Comparing coefficients of x A + B = 0 ....(iv)

Comparing constants B = 1

On solving eq. (iii), (iv) and (v), we get A = -1, B = 1, C = 1

Putting these values of A, B and C in eq. (ii),

$$\frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}$$

$\therefore$ I = $\int_1^3 \left(\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right) dx$

= $\int_1^3 \left(\frac{-1}{x}\right) dx + \int_1^3 \left(\frac{1}{x^2}\right) dx + \int_1^3 \left(\frac{1}{x+1}\right) dx$

$= -(\log|x|)_1^3 + \int_1^3 x^{-2} dx + (\log|x+1|)_1^3$

= $-(\log|3| - \log|1|) + \left(\frac{x^{-1}}{-1}\right)_1^3 + (\log|4| - \log|2|)$

$\Rightarrow$ I = $-\log 3 + 0 - \left(\frac{1}{x}\right)_1^3 + (\log 4 - \log 2)$

= $-\log 3 - \left(\frac{1}{3} - 1\right) + (\log 2^2 - \log 2)$

$\Rightarrow$ I = $-\log 3+\frac{2}{3}+2\log 2-\log 2$

= $-\log 3+\frac{2}{3}+\log 2$

= $\frac{2}{3}+\log 2-\log 3$

= $\frac{2}{3}+\log\frac{2}{3}$ = LHS Hence Proved

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35. $\int_0^1 xe^x\ dx=1$

Ans. Taking RHS

Let I = $\int_0^1 xe^x\ dx$

[Applying Product rule]

= $\left(xe^x\right)_0^1-\int_0^1 1.e^x\ dx$

$\Rightarrow$ I = $e-0-\int_0^1 e^x\ dx$

= $e-\left(e^x\right)_0^1$

= $e-\left(e-e^0\right)$

= 1 = LHS Hence Proved

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36. $\int_{-1}^{1} x^{17} \cos^4 x \, dx = 0$

Ans. Let I = $\int_{-1}^{1} x^{17} \cos^4 x \, dx$

Here $f(x) = x^{17} \cos^4 x$

$\Rightarrow f(-x) = (-x)^{17} \cos^4 (-x)$

= $-x^{17} \cos^4 x = -f(x)$

$\therefore$ $f(x)$ is an odd function of x.

$\left[\because \int_{-a}^{a} f(x) \, dx = 0, \text{ if } f(x) \text{ is ann odd funtion of } x \right]$

= 0 = LHS Hence proved

37. $\int_{0}^{\frac{\pi}{2}} \sin^3 x \, dx = \frac{2}{3}$

Ans. Let I = $\int_{0}^{\frac{\pi}{2}} \sin^3 x \, dx$

= $\int_{0}^{\frac{\pi}{2}} \frac{1}{4}(3\sin x - \sin 3x) \, dx$

= $\frac{1}{4}\left[3(-\cos x) - \left(\frac{-\cos 3x}{3} \right) \right]_{0}^{\frac{\pi}{2}}$

$= \frac{1}{4}\left[-3\cos x + \frac{1}{3}\cos 3x\right]_0^{\frac{\pi}{2}}$

= $\frac{1}{4}\left[\left(-3\cos\frac{\pi}{2} + \frac{1}{3}\cos\frac{3\pi}{2}\right) - \left(-3\cos 0 + \frac{1}{3}\cos 0\right)\right]$

$\Rightarrow$ I = $\frac{1}{4}\left[-3\times 0 + \frac{1}{3}\times 0 + 3\times 1 - \frac{1}{3}\times 1\right]$

= $\frac{1}{4}\left(3 - \frac{1}{3}\right)$

= $\frac{1}{4}\times\frac{8}{3} = \frac{2}{3}$ = LHS Hence proved

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38. $\int_0^{\frac{\pi}{4}} 2\tan^3 x\, dx = 1 - \log 2$

Ans. Let I = $\int_0^{\frac{\pi}{4}} 2\tan^3 x\, dx$

= $2\int_0^{\frac{\pi}{4}} \tan x \tan^2 x\, dx$

= $2\int_0^{\frac{\pi}{4}} \tan x\left(\sec^2 x - 1\right)\, dx$

$\Rightarrow$ I = $2\int_0^{\frac{\pi}{4}} \left(\tan x \sec^2 x - \tan x\right)\, dx$

= $2\left[\int_0^{\frac{\pi}{4}}\left(\tan x \sec^2 x\right)\ dx - \int_0^{\frac{\pi}{4}} \tan x\ dx\right]$..........(i)

Let $I_1 = \int_0^{\frac{\pi}{4}}\left(\tan x.\sec^2 x\right)\ dx$

Putting $\tan x = t$

$\Rightarrow \sec^2\theta\ dx = dt$

Limits of integration when $x = 0, t = \tan 0 = 0$ and when $x = \frac{\pi}{4}, t = \tan\frac{\pi}{4} = 1$

$\therefore$ $I_1 = \int_0^1 t\ dt$

= $\left(\frac{t^2}{2}\right)_0^1$

= $\frac{1}{2} - 0 = \frac{1}{2}$

Putting value of I_1 in eq. (i),

$I = 2\left[\frac{1}{2} - \int_0^{\frac{\pi}{4}} \tan x\ dx\right]$

$= 2\left[\frac{1}{2} - (\log\sec x)_0^{\frac{\pi}{2}}\right]$

= $1 - 2\left(\log\sec\frac{\pi}{4} - \log\sec 0\right)$

$\Rightarrow$ I = $1-2\left(\log\sqrt{2}-\log 1\right)$

= $1-2\left(\log 2^{\frac{1}{2}}-0\right)$

= $1-2\left(\frac{1}{2}\log 2\right)$

= $1-\log 2$ = LHS Hence Proved

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39. $\int_0^1 \sin^{-1} x\, dx = \frac{\pi}{2}-1$

Ans. Let I = $\int_0^1 \sin^{-1} x\, dx$

Putting $x = \sin\theta$

$\Rightarrow dx = \cos\theta\, d\theta$

Limits of integration when $x=0, \theta=0$ and when $x=1, \sin\theta=1$, i.e., $\theta=\frac{\pi}{2}$

$\therefore$ I = $\int_0^1 \sin^{-1} x\, dx$

= $\int_0^1 \theta\cos\theta\, d\theta$

[Integrating by parts]

= $\left[\theta\sin\theta\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 1.\sin\theta\, d\theta$

$\Rightarrow$ I = $\left(\frac{\pi}{2}-0\right)+\left[\cos\theta\right]_0^{\frac{\pi}{2}}$

= $\frac{\pi}{2}+\left(\cos\frac{\pi}{2}-\cos 0\right)$

= $\frac{\pi}{2}+(0-1)$

= $\frac{\pi}{2}-1$ = LHS Hence proved

40. Evaluate $\int_0^1 e^{2-3x}\ dx$ as a limit of sum.

Ans. Given: $\int_0^1 e^{2-3x}\ dx$

Comparing $\int_a^b f(x)\ dx$ we have, $a=0, b=1, f(x)=e^{2-3x}$

$\therefore\ nh=b-a=1$

Putting these values in

$=>\int_a^b f(x)dx=\lim_{h\to 0}[f(a)+f(a+h)+f(a+2h)+\ldots+f(a+(n-1)h)]$

$=\lim_{h\to 0}[e^2+e^{2-3h}+e^{2-6h}+\ldots+e^{2+3(n-1)h}]$

$=e^2\lim_{h\to 0}h[\frac{e^{-3nh}-1}{e^{-3h}-1}]$

= $=e^2\lim_{h\to 0}h[\frac{e^{-3}-1}{e^{-3h}-1}]$

$=\lim_{h\to 0}\frac{-3h}{e^{-3h}-1}\times\frac{-1}{3}\times(e^2)(e^{-3}-1)$

$\Rightarrow$ I = $\left(e^{-1}-e^{2}\right)\times 1\times\frac{-1}{3}$

= $\frac{1}{3}\left(e^{2}-\frac{1}{e}\right)$ = LHS Hence Proved

41. Choose the correct answer:

$\int\frac{dx}{e^{x}+e^{-x}}$ **is equal to:**

(A) $\tan^{-1}\left(e^{x}\right)+c$

(B) $\tan^{-1}\left(e^{-x}\right)+c$

(C) $\log\left(e^{x}-e^{-x}\right)+c$

(D) $\log\left(e^{x}+e^{-x}\right)+c$

Ans. Let I = $\int\frac{dx}{e^{x}+e^{-x}}$

= $\int\frac{1}{e^{x}+\frac{1}{e^{x}}}\,dx$

= $\int\frac{1}{\left(\frac{e^{2x}+1}{e^{x}}\right)}\,dx$

= $\int\frac{e^{x}}{e^{2x}+1}\,dx$..........(i)

Putting $e^{x}=t$

$\Rightarrow e^{x}\,dx=dt$

$\therefore$ From eq. (i), I = $\int \frac{dt}{t^2+1}$

= $\tan^{-1} t + c$

= $\tan^{-1} e^x + c$

Therefore, option (A) is correct.

42. Choose the correct answer:

$\int \frac{\cos 2x}{(\sin x + \cos x)^2}\, dx$ **is equal to:**

(A) $\frac{-1}{\sin x + \cos x} + c$

(B) $\log |\sin x + \cos x| + c$

(C) $\log |\sin x - \cos x| + c$

(D) $\frac{1}{(\sin x + \cos x)^2}$

Ans. Let I = $\int \frac{\cos 2x}{(\sin x + \cos x)^2}\, dx$

= $\int \frac{\cos^2 x - \sin^2 x}{(\sin x + \cos x)^2}\, dx$

= $\int \frac{(\cos x + \sin x)(\cos x - \sin x)}{(\sin x + \cos x)(\sin x + \cos x)}\, dx$

$\Rightarrow$ I = $\int \frac{\cos x - \sin x}{\sin x + \cos x}\, dx$

= $\log|\sin x+\cos x|+c$

Therefore, option (B) is correct.

43. Choose the correct answer:

If $f(a+b-x)=f(x)$, **then** $\int_a^b xf(x)\, dx$ **is equal to:**

(A) $\frac{a+b}{2}\int_a^b f(b-x)\, dx$

(B) $\frac{a+b}{2}\int_a^b f(b+x)\, dx$

(C) $\frac{b-a}{2}\int_a^b f(x)\, dx$

(D) $\frac{a+b}{2}\int_a^b f(x)\, dx$

Ans. Given: $f(a+b-x)=f(x)$ ....(i)

Let I = $\int_a^b xf(x)\, dy$ ....(ii)

$\Rightarrow$ I = $\int_a^b (a+b-x)f(a+b-x)\, dx$

= $\int_a^b (a+b-x)f(x)\, dx$ ....(iii)

Adding eq. (ii) and (iii),

2I = $\int_a^b (x+a+b-x) f(x)\ dx$

= $\int_a^b (a+b) f(x)\ dx$

= $(a+b)\int_a^b f(x)\ dx$

$\Rightarrow$ I = $\left(\frac{a+b}{2}\right)\int_a^b f(x)\ dx$

Therefore, option (D) is correct.

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44. The value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx$ is:

(A) 1

(B) 0

(C) -1

(D) $\frac{\pi}{4}$

Ans. Let I = $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx$

= $\int_0^1 \tan^{-1}\left(\frac{x+x-1}{1+x-x^2}\right) dx$

= $\int_0^1 \tan^{-1}\left(\frac{x+(x-1)}{1-x(x-1)}\right) dx$

$$\Rightarrow \mathrm{I} = \int_0^1 \tan^{-1} x + \tan^{-1}(x-1)\, dx$$

$$= \int_0^1 \tan^{-1} x\, dx + \int \tan^{-1}(x-1)\, dx$$

$$\Rightarrow \mathrm{I} = \int_0^1 \tan^{-1} x\, dx + \int \tan^{-1}(1-x-1)\, dx$$

$$= \int_0^1 \tan^{-1} x\, dx + \int \tan^{-1}(-x)\, dx$$

$$\Rightarrow \mathrm{I} = \int_0^1 \tan^{-1} x\, dx - \int \tan^{-1} x\, dx = 0$$

Therefore, option (B) is correct.