CBSE Class–12 Mathematics

NCERT solution

Chapter - 7

Integrals - Exercise 7.3

Find the integrals of the following functions in Exercises 1 to 9.

1. $\sin^2(2x+5)$

Ans. $\int \sin^2(2x+5)\ dx$

$= \int \frac{1}{2}\{1-\cos 2(2x+5)\}\ dx$

Using $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$= \frac{1}{2}\int\{1-\cos(4x+10)\}\ dx$

$= \frac{1}{2}\left[\int 1\ dx - \int \cos(4x+10)\ dx\right]$

Using $\int \cos(ax+b)dx = \frac{\sin(ax+b)}{a} + c$

$= \frac{1}{2}\left[x - \frac{\sin(4x+10)}{4}\right] + c$

$= \frac{1}{2}x - \frac{1}{8}\sin(4x+10) + c$ Ans.

2. $\sin 3x \cos 4x$

Ans. $\int \sin 3x \cos 4x\ dx = \frac{1}{2}\int 2\sin 3x \cos 4x\ dx$

$= = \frac{1}{2}\int\{\sin(4x+3x) - \sin(4x-3x)\}\ dx$ Using 2sinB cosA=sin(A+B)-sin(A-B)

= $\frac{1}{2}\int(\sin 7x-\sin x)\ dx$

= $\frac{1}{2}\left[\int \sin 7x\ dx-\int \sin x\ dx\right]$

= $\frac{1}{2}\left[\frac{-\cos 7x}{7}-(-\cos x)\right]+c$

= $\frac{-1}{14}\cos 7x+\frac{1}{2}\cos x+c$ Ans.

3. $\cos 2x\cos 4x\cos 6x$

Ans. $\int \cos 2x\cos 4x\cos 6x\ dx$

= $\frac{1}{2}\int 2(\cos 6x\cos 4x)\cos 2x\ dx$

= $\frac{1}{2}\left[\int \{\cos 10x+\cos 2x\}\cos 2x\ dx\right]$

Using 2cosA cosB=cos(A+B)+cos(A-B)

= $\frac{1}{2}\left[\int (\cos 10x\cos 2x+\cos^2 2x)\ dx\right]$

= $\frac{1}{4}\left[\int \{(2\cos 10x\cos 2x)+2\cos^2 2x\}\ dx\right]$

= $\frac{1}{4}\left[\int (\cos 12x+\cos 8x+1+\cos 4x)\ dx\right]$

Using 2cosA cosB=cos(A+B)+cos(A-B)

and $2\cos^2\theta = 1+\cos 2\theta$

= $\frac{1}{4}\left[\int \cos 12x\ dx+\int \cos 8x\ dx+\int \cos 4x\ dx+\int 1\ dx\right]$

$= \frac{1}{4}\left(\frac{\sin 12x}{12} + \frac{\sin 8x}{8} + \frac{\sin 4x}{4} + x\right) + c$ Ans.

4. $\sin^3(2x+1)$

Ans. $\int \sin^3(2x+1)\ dx$

$= \int\left(\frac{3}{4}\sin(2x+1) - \frac{1}{4}\sin 3(2x+1)\right) dx$

$\left[\because \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta\right]$

$= \int\left(\frac{3}{4}\sin(2x+1) - \frac{1}{4}\sin(6x+3)\right) dx$

$= \frac{3}{4}\int \sin(2x+1)\ dx - \frac{1}{4}\int \sin(6x+3)\ dx$

$= \frac{3}{4}\left(\frac{-\cos(2x+1)}{2}\right) - \frac{1}{4}\left(\frac{-\cos(6x+3)}{6 \rightarrow \text{Coeff. of } x}\right) + c$

$= \frac{-3}{8}\cos(2x+1) + \frac{1}{24}\cos(6x+3) + c$

<u>Another Method</u> $\int \sin^3(2x+1)\ dx$

$= \int \text{Sin}^2(2x+1)\,\text{Sin}(2x+1)dx$

$= \int\left(1 - \cos^2(2x+1)\right)\text{Sin}(2x+1)dx$

Using $\sin^2\theta = 1 - \cos^2\theta$

let $\cos(2x+1) = t$

therefore $-2\sin(2x+1)dx = dt$

thus $\sin(2x+1)dx = -\frac{1}{2}dt$

$= -\frac{1}{2}\int\left(1-t^2\right)dt$

$= -\frac{1}{2}\left(\int 1dt - \int t^2 dt\right)$

$= -\frac{1}{2}\left(t - \frac{t^3}{3}\right) + c$

$= -\frac{1}{2}\text{Cos}\left(2x+1\right) + \frac{1}{6}\text{Cos}^3\left(2x+1\right) + c$ Ans

5. $\sin^3 x \cos^3 x$

Ans. $\int \sin^3 x \cos^3 x \, dx$

$= \int (\sin x \cos x)^3 \, dx$

$= \frac{1}{8}\int (2\sin x \cos x)^3 \, dx$

$= \frac{1}{8}\int (\sin 2x)^3 \, dx$

$= \frac{1}{8}\int \sin^3 2x \, dx$

$= \frac{1}{8}\int \left(\frac{3}{4}\sin 2x - \frac{1}{4}\sin 6x\right) dx$

$= \frac{3}{32}\int \sin 2x \, dx - \frac{1}{32}\int \sin 6x \, dx$

$= \frac{-3}{32}\frac{\cos 2x}{2} - \frac{1}{32}\left(\frac{-\cos 6x}{6}\right) + c$

$= \frac{-3}{64}\cos 2x + \frac{1}{192}\cos 6x + c$

Another method let I= $\int \sin^3 x \cos^3 x \, dx$

$I = \int (1 - \cos^2 x) \cos^3 x \sin x dx$

$= \int (\cos^3 x - \cos^5 x) \sin x dx$

let cosx=t

-sinx dx=dt

sinx dx = -dt

$I = -\int (t^3 - t^5) dt$

$I = \int (t^5 - t^3) dt$

$= \frac{t^6}{6} - \frac{t^4}{4} + c$

$= \frac{\cos^6}{6} - \frac{\cos^4}{4} + c$ ans.

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6. $\sin x \sin 2x \sin 3x$

Ans. $\int \sin x \sin 2x \sin 3x \, dx$

= $\frac{1}{2} \int (2 \sin 3x \sin 2x) \sin x \, dx$

= $\frac{1}{2} \int (\cos x - \cos 5x) \sin x \, dx$ Using 2sin A sin B=cos (A-B) - cos (A+B)

$= \frac{1}{2} \int (\cos x \sin x - \cos 5x \sin x) dx)$

$= \frac{1}{4} \int (2 \cos x \sin x - 2 \cos 5x \sin x) dx)$

=$= \frac{1}{4} \int (2 \sin x \cos x - \{\sin(5x + x) - \sin(5x - x)\}) \, dx$

using 2cosA sinB=sin(A+B)-sin(A-B)

= $\frac{1}{4}\int(\sin 2x - \sin 6x + \sin 4x)\ dx$

= $\frac{1}{4}\left[\int \sin 2x\ dx + \int \sin 4x\ dx - \int \sin 6x\ dx\right]$

as $\int \sin(ax+b)dx = \frac{-\cos(ax+b)}{a} + c$

= $\frac{1}{4}\left(\frac{-\cos 2x}{2} - \frac{\cos 4x}{4} + \frac{\cos 6x}{6}\right) + c$ Ans

7. $\sin 4x \sin 8x$

Ans. $\int \sin 4x \sin 8x\ dx$

= $\frac{1}{2}\int 2\sin 4x \sin 8x\ dx$ Using 2sinA sinB= cos(A-B)-cos(A+B)

$= \frac{1}{2}\int(\cos(4x-8x) - \cos(4x+8x))\,dx$

= $\frac{1}{2}\int(\cos(-4x) - \cos 12x)\ dx$

= $\frac{1}{2}\int(\cos 4x - \cos 12x)\ dx$

As cos(-x)=cosx

= $\frac{1}{2}\left[\int \cos 4x\ dx - \int \cos 12x\ dx\right]$

= $\frac{1}{2}\left(\frac{\sin 4x}{4} - \frac{\sin 12x}{12}\right) + c$ Ans. As $\int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a} + c$

8. $\frac{1-\cos x}{1+\cos x}$

Ans. $\int \frac{1-\cos x}{1+\cos x}\, dx$

$= \int \frac{2\sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}}\, dx$

As $\begin{array}{l} 2\sin^2\frac{\theta}{2} = 1-\cos\theta \\ 2\cos^2\frac{\theta}{2} = 1+\cos\theta \end{array}$

$= \int \tan^2 \frac{x}{2}\, dx$

$= \int \left(\sec^2 \frac{x}{2} - 1\right) dx$

$= \int \sec^2 \frac{x}{2}\, dx - \int 1\, dx$

As $\int \sec^2(ax+b)\,dx = \frac{\tan(ax+b)}{a} + c$

$= 2\tan\frac{x}{2} - x + c$ Ans.

9. $\frac{\cos x}{1+\cos x}$

Ans. $\int \frac{\cos x}{1+\cos x}\, dx$

$= \int \frac{1+\cos x - 1}{1+\cos x}\, dx$

= $\int \frac{1+\cos x}{1+\cos x} - \frac{1}{1+\cos x}\ dx$

= $\int \left(1 - \frac{1}{2\cos^2\frac{x}{2}}\right) dx$

As $2\cos^2\frac{\theta}{2} = 1 + \cos\theta$

= $\int 1\, dx - \frac{1}{2}\int \sec^2\frac{x}{2}\ dx$

as $\int \sec^2(ax+b)\,dx = \frac{\tan(ax+b)}{a} + c$

= $x - \frac{1}{2}\frac{\tan\frac{x}{2}}{\frac{1}{2}} + c$

= $x - \tan\frac{x}{2} + c$ Ans.

Find the integrals of the following functions in Exercises 10 to 18.

10. $\sin^4 x$

Ans. $\int \sin^4 x\, dx$

= $\int \left(\sin^2 x\right)^2\ dx$

= $\int \left(\frac{1-\cos 2x}{2}\right)^2\ dx$

as $2\sin^2\theta = 1 - \cos 2\theta$

= $\int \frac{(1-\cos 2x)^2}{4}\ dx$

= $\frac{1}{4}\int \left(1+\cos^2 2x-2\cos 2x\right)\ dx$

= $\frac{1}{4}\int \left\{1+\left(\frac{1+\cos 4x}{2}\right)-2\cos 2x\right\}\ dx$

as $2\cos^2 2\theta = 1+\cos 4\theta$

= $\frac{1}{4}\int \left(\frac{2+1+\cos 4x-4\cos 2x}{2}\right)\ dx$

= $\frac{1}{8}\int \left(3+\cos 4x-4\cos 2x\right)\ dx$

= $\frac{1}{8}\left[3\int 1\ dx+\int \cos 4x\ dx-4\int \cos 2x\ dx\right]$

= $\frac{1}{8}\left[3x+\frac{\sin 4x}{4}-\frac{4\sin 2x}{2}\right]+c$

As $\int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a}+c$

= $\frac{3}{8}x+\frac{1}{32}\sin 4x-\frac{1}{4}\sin 2x+c$ Ans

11. $\cos^4 2x$

Ans. $\int \cos^4 2x\ dx$

= $\int \left(\cos^2 2x\right)^2\ dx$

= $\int \left(\frac{1+\cos 4x}{2}\right)^2\ dx$

As $2\cos^2 2\theta = 1 + \cos 4\theta$

= $\int \frac{(1+\cos 4x)^2}{4}\ dx$

= $\frac{1}{4}\int \left(1+\cos^2 4x+2\cos 4x\right)\ dx$

= $\frac{1}{4}\int \left\{1+\left(\frac{1+\cos 8x}{2}\right)+2\cos 4x\right\}\ dx$

as $2\cos^2 4\theta = 1 + \cos 8\theta$

= $\frac{1}{4}\int \left(\frac{2+1+\cos 8x+4\cos 4x}{2}\right) dx$

= $\frac{1}{8}\int \left(3+\cos 8x+4\cos 4x\right)\ dx$

= $\frac{1}{8}\left[3\int 1\ dx+\int \cos 8x\ dx+4\int \cos 4x\ dx\right]$

= $\frac{1}{8}\left[3x+\frac{\sin 8x}{8}+\frac{4\sin 4x}{4}\right]+c$

using $\int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a} + c$

= $\frac{3}{8}x+\frac{1}{64}\sin 8x+\frac{1}{8}\sin 4x+c$ Ans.

12. $\frac{\sin^2 x}{1+\cos x}$

Ans. $\int \frac{\sin^2 x}{1+\cos x}\ dx$

= $\int \frac{1-\cos^2 x}{1+\cos x}\ dx$

= $\int \frac{(1-\cos x)(1+\cos x)}{1+\cos x}\ dx$

= $\int (1-\cos x)\ dx$

= $\int 1\ dx - \int \cos x\ dx$

= $x - \sin x + c$ Ans.

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13. $\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$

Ans. $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}\ dx$

= $\int \frac{(2\cos^2 x - 1) - (2\cos^2 \alpha - 1)}{\cos x - \cos \alpha}\ dx$

using $\cos 2\theta = 2\cos^2\theta - 1$

= $\int \frac{2\cos^2 x - 1 - 2\cos^2 \alpha + 1}{\cos x - \cos \alpha}\ dx$

= $\int \frac{2\cos^2 x - 2\cos^2 \alpha}{\cos x - \cos \alpha}\ dx$

= $2\int \frac{\cos^2 x - \cos^2 \alpha}{\cos x - \cos \alpha}\ dx$

= $2\int \frac{(\cos x - \cos \alpha)(\cos x + \cos \alpha)}{(\cos x - \cos \alpha)}\ dx$

= $2\int (\cos x + \cos \alpha)\ dx$

= $2\left[\int \cos x\, dx + \int \cos\alpha\, dx\right]$

= $2\left[\sin x + \cos\alpha \int 1\right] + c$

= $2\left[\sin x + \cos\alpha\,(x)\right] + c$

= $2\sin x + 2x\cos\alpha + c$

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14. $\dfrac{\cos x - \sin x}{1+\sin 2x}$

Ans. Let I = $\int \dfrac{\cos x - \sin x}{1+\sin 2x}\, dx$

= $\int \dfrac{\cos x - \sin x}{\cos^2 x + \sin^2 x + 2\sin x \cos x}\, dx$

using identity $1 = \sin^2\theta + \cos^2\theta$

= $\int \dfrac{\cos - \sin x}{(\cos x + \sin x)^2}\, dx$ .....(i)

Putting $\cos x + \sin x = t$

$\Rightarrow -\sin x + \cos x = \dfrac{dt}{dx}$

$\Rightarrow (\cos x - \sin x)\, dx = dt$

$\therefore$ From eq. (i), I = $\int \dfrac{dt}{t^2}$

= $\int t^{-2}\, dt$

= $\dfrac{t^{-1}}{-1} + c$

= $\dfrac{-1}{t}+c$

= $\dfrac{-1}{\cos x+\sin x}+c$

15. $\tan^3 2x\sec 2x$

Ans. Let I = $\int \tan^3 2x\sec 2x\,dx$

= $\int \tan^2 2x\tan 2x\sec 2x\,dx$

= $\int \left(\sec^2 2x-1\right)\tan 2x\sec 2x\,dx$

= $\dfrac{1}{2}\int \left(\sec^2 2x-1\right)\left(2\sec 2x\tan 2x\right)\,dx$

Putting $\sec 2x = t$

$\Rightarrow \sec 2x\tan 2x\dfrac{d}{dx}(2x)=\dfrac{dt}{dx}$

$\Rightarrow 2\sec 2x\tan 2x\,dx = dt$

$\therefore$ From eq. (i),

I = $\dfrac{1}{2}\int \left(t^2-1\right)\,dt$

= $\dfrac{1}{2}\left(\int t^2\,dt-\int 1\,dt\right)$

= $\dfrac{1}{2}\left(\dfrac{t^3}{3}-t\right)+c$

= $\dfrac{1}{6}t^3-\dfrac{1}{2}t+c$

= $\frac{1}{6}\sec^3 2x - \frac{1}{2}\sec 2x + c$

16. $\tan^4 x$

Ans. $\int \tan^4 x\, dx$

= $\int \tan^2 x \tan^2 x\, dx$

= $\int \tan^2 x\left(\sec^2 x - 1\right)\, dx$

= $\int \tan^2 x \sec^2 x\, dx - \int\left(\sec^2 x - 1\right)\, dx$

= $\int \tan^2 x \sec^2 x\, dx - \int \sec^2 x\, dx + \int 1\, dx$..........(i)

Putting $\tan x = t$

$\Rightarrow \sec^2 x = \frac{dt}{dx}$

$\Rightarrow \sec^2 x\, dx = dt$

$\therefore$ From eq. (i),

= $\int t^2\, dt - \tan x + x + c$

= $\frac{t^3}{3} - \tan x + x + c$

= $\frac{1}{3}\tan^3 x - \tan x + x + c$

17. $\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$

Ans. $\int \frac{\sin^3 x+\cos^3 x}{\sin^2 x\cos^2 x}\ dx$

$= \int \frac{\sin^3 x}{\sin^2 x\cos^2 x}+\frac{\cos^3 x}{\sin^2 x\cos^2 x}\ dx$

$= \int \frac{\sin x}{\cos^2 x}+\frac{\cos x}{\sin^2 x}\ dx$

$= \int \frac{\sin x}{\cos x\cos x}+\frac{\cos x}{\sin x\cos x}\ dx$

$= \int \left(\tan x\sec x+\cot x\cos ec\ x\right)\ dx$

$= \int \tan x\sec x\ dx+\int \cos ec\ x\cot x\ dx$

$= \sec x-\cos ec\ x+c$

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18. $\frac{\cos 2x+2\sin^2 x}{\cos^2 x}$

Ans. $\int \frac{\cos 2x+2\sin^2 x}{\cos^2 x}\ dx$

$= \int \frac{\left(1-2\sin^2 x\right)+2\sin^2 x}{\cos^2 x}\ dx$

using $\cos 2\theta = 1 - 2\sin^2\theta$

$= \int \frac{1}{\cos^2 x}\ dx$

$= \int \sec^2 x\ dx$

$= \tan x+c$

Integrate the following functions in Exercises 19 to 22.

19. $\dfrac{1}{\sin x \cos^3 x}$

Ans. Let I = $\int \dfrac{1}{\sin x \cos^3 x}\, dx$ ....(i)

Dividing numerator and denominator by $\cos^4\theta$

$= \int \frac{1/\cos^4 x dx}{\sin x / \cos x}$

$I = \int \frac{\sec^4 x dx}{\tan x}$

= $\int \dfrac{\sec^2 x \sec^2 x}{\tan x}\, dx$

$I = \int \frac{(1+\tan^2 x)\sec^2 x dx}{\tan x}$ ....(ii)

Putting $\tan x = t$

$\Rightarrow \sec^2 x = \dfrac{dt}{dx}$

$\Rightarrow \sec^2 x\, dx = dt$

$\therefore$ From eq. (ii),

I = $\int \dfrac{(1+t^2)}{t}\, dt$

= $\int \left(\dfrac{1}{t} + \dfrac{t^2}{t}\right) dt$

= $\int \left(\dfrac{1}{t} + t\right) dt$

= $\int \frac{1}{t}\ dt + \int t\ dt$

= $\log|t| + \frac{t^2}{2} + c$

= $\log|\tan x| + \frac{1}{2}\tan^2 x + c$

20. $\frac{\cos 2x}{(\cos x + \sin x)^2}$

Ans. Let I = $\int \frac{\cos 2x}{(\cos x + \sin x)^2}\ dx$

= $\int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2}\ dx$

= $\int \frac{(\cos x + \sin x)(\cos x - \sin x)}{(\cos x + \sin x)(\cos x + \sin x)}\ dx$

= $\int \frac{(\cos x - \sin x)}{(\cos x + \sin x)}\ dx$ ....(i)

Putting $\cos x + \sin x = t$

$\Rightarrow -\sin x + \cos x = \frac{dt}{dx}$

$\Rightarrow (\cos x - \sin x)\ dx = dt$

$\therefore$ From eq. (i),

I = $\int \frac{dt}{t}$

= $\log|t|+c$

= $\log|\cos x+\sin x|+c$

21. $\sin^{-1}(\cos x)$

Ans. $\int \sin^{-1}(\cos x)\ dx$

= $\int \sin^{-1}\sin\left(\frac{\pi}{2}-x\right)\ dx$

= $\int\left(\frac{\pi}{2}-x\right)\ dx$

= $\int\frac{\pi}{2}\ dx-\int x\ dx$

= $\frac{\pi}{2}\int 1\ dx-\int x\ dx$

= $\frac{\pi}{2}x-\frac{x^2}{2}+c$

22. $\frac{1}{\cos(x-a)\cos(x-b)}$

Ans. Let I = $\int\frac{1}{\cos(x-a)\cos(x-b)\ dx}$ ....(i)

Multiplying and dividing by

$\sin(b-a)$ as $(x-a)-(x-b)=b-a$,

I = $\frac{1}{\sin(b-a)}\int\frac{\sin(b-a)}{\cos(x-a)\cos(x-b)}\ dx$

$= \frac{1}{\sin(b-a)}\int \frac{\sin[(x-a)-(x-b)]}{\cos(x-a)\cos(x-b)}\,dx$

$= \frac{1}{\sin(b-a)}\int \frac{\sin(x-a)\cos(x-b)-\cos(x-a)\sin(x-b)}{\cos(x-a)\cos(x-b)}\,dx$

using sin(A-B)=sinAcosB-cosAsinB

$= \frac{1}{\sin(b-a)}\int \left[\frac{\sin(x-a)\cos(x-b)}{\cos(x-a)\cos(x-b)}-\frac{\cos(x-a)\sin(x-b)}{\cos(x-a)\cos(x-b)}\right]dx$

$= \frac{1}{\sin(b-a)}\int [\tan(x-a)-\tan(x-b)]\,dx$

$= \frac{1}{\sin(b-a)}\left[-\log|\cos(x-a)|+\log|\cos(x-b)|\right]+c$

$= \frac{1}{\sin(b-a)}\log\left|\frac{\cos(x-b)}{\cos(x-a)}\right|+c$

Choose the correct answer in Exercise 23 and 24.

23. $\int \frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}\,dx$ **is equal to:**

(A) $\tan x+\cot x+C$

(B) $\tan x+\operatorname{cos}ec\, x+C$

(C) $-\tan x+\cot x+C$

(D) $\tan x+\sec x+C$

Ans. $\int \frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}\,dx$

= $\int \frac{\sin^2 x}{\sin^2 x \cos^2 x} - \frac{\cos^2 x}{\sin^2 x \cos^2 x}\ dx$

= $\int \frac{1}{\cos^2 x} - \frac{1}{\sin^2 x}\ dx$

= $\int \sec^2 x - \cos ec^2 x\ dx$

= $\int \sec^2 x\ dx - \int \cos ec^2 x\ dx$

= $\tan x - (-\cot x) + C$

= $\tan x + \cot x + C$

Therefore, option (A) is correct.

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24. $\int \frac{e^x(1+x)}{\cos^2(e^x x)}\ dx$ **is equal to:**

(A) $-\cot(ex^x) + C$

(B) $\tan(xe^x) + C$

(C) $\tan(e^x) + C$

(D) $\cot(e^x) + C$

Ans. Let I = $\int \frac{e^x(1+x)}{\cos^2(e^x x)}\ dx$ ....(i)

Putting $e^x\ x = t$

$\Rightarrow\ e^x.1 + xe^x = \frac{dt}{dx}$

$\Rightarrow e^x(1+x)\ dx = dt$

$\therefore$ From eq. (i),

I = $\int \frac{dt}{\cos^2 t}$

= $\int \sec^2 t\ dt$

= $\tan t + C$

= $\tan\left(x.e^x\right) + C$

Therefore, option (B) is correct.