CBSE Class–12 Mathematics

NCERT solution

Chapter - 7

Integrals - Exercise 7.5

Integrate the (rational) function in Exercises 1 to 6.

1. $\frac{x}{(x+1)(x+2)}$

Ans. $\frac{x}{(x+1)(x+2)}=\frac{A}{x+1}+\frac{B}{x+2}$ ....(i)

$\Rightarrow\ x=A(x+2)+B(x+1)$

$\Rightarrow\ x=Ax+2A+Bx+B$

Comparing coefficients of x on both sides A + B = 1 .....(ii)

Comparing constants 2A + B = 0 .....(iii)

Solving eq. (ii) and (iii), we get A = -1 and B = 2

Putting these values of A and B in eq. (i),

$$\frac{x}{(x+1)(x+2)}=\frac{-1}{x+1}+\frac{2}{x+2}$$

$$\therefore\ \int\frac{x}{(x+1)(x+2)}\ dx=-\int\frac{1}{x+1}\ dx+2\int\frac{1}{x+2}\ dx$$

$$=-log|x+1|+2log|x+2|+c$$

$$=\log|x+2|^2-\log|x+1|+c$$

= $\log \frac{(x+2)^2}{|x+1|} + c$

2. $\frac{1}{x^2-9}$

Ans. $\int \frac{1}{x^2-9}\ dx$

= $\int \frac{1}{x^2-3^2}\ dx$

= $\frac{1}{2\times 3}\log\left|\frac{x-3}{x+3}\right| + c$

$\left[\because \int \frac{1}{x^2-a^2}\ dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$

= $\frac{1}{6}\log\left|\frac{x-3}{x+3}\right| + c$

3. $\frac{3x-1}{(x-1)(x-2)(x-3)}$

Ans. $\frac{3x-1}{(x-1)(x-2)(x-3)}$

= $\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$ ....(i)

$\Rightarrow 3x-1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$

$\Rightarrow 3x-1 = A(x^2-5x+6) + B(x^2-4x+3) + C(x^2-3x+2)$

$\Rightarrow 3x-1 = Ax^2 - 5Ax + 6A + Bx^2 - 4Bx + 3B + Cx^2 - 3Cx + 2C$

Comparing coefficients of x^2 : A + B + C = 0 ....(ii)

Comparing coefficients of x : –5A – 4B – 3C= 3

$\Rightarrow$ 5A + 4B + 3C = –3 …….(iii)

Comparing constants: 6A + 3B + 2C = –1 …….(iv)

On solving eq. (i), (ii) and (iii), we get A = 1, B = –5, C = 4

Putting the values of A, B and C in eq. (i),

$$\frac{3x-1}{(x-1)(x-2)(x-3)}$$

$$= \frac{1}{x-1} + \frac{-5}{x-2} + \frac{4}{x-3}$$

$$\Rightarrow \int \frac{3x-1}{(x-1)(x-2)(x-3)}\, dx$$

$$= \int \frac{1}{x-1}\, dx - 5\int \frac{1}{x-2}\, dx + 4\int \frac{1}{x-3}\, dx$$

$$\Rightarrow \int \frac{3x-1}{(x-1)(x-2)(x-3)}\, dx$$

$$= \log|x-1| - 5\log|x-2| + 4\log|x-3| + c$$

4. $\frac{x}{(x-1)(x-2)(x-3)}$

Ans. $\frac{x}{(x-1)(x-2)(x-3)}$

$$= \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} \quad \text{…….(i)}$$

$$\Rightarrow x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

$\Rightarrow\ x = A\left(x^2-5x+6\right)+B\left(x^2-4x+3\right)+C\left(x^2-3x+2\right)$

$\Rightarrow\ x = Ax^2-5Ax+6A+Bx^2-4Bx+3B+Cx^2-3Cx+2C$

Comparing coefficients of x^2 : A + B + C = 0 ....(ii)

Comparing coefficients of x : –5A – 4B – 3C= 1

$\Rightarrow$ 5A + 4B + 3C = –1 ....(iii)

Comparing constants: 6A + 3B + 2C = 0 ....(iv)

On solving eq. (i), (ii) and (iii), we get A = $\frac{1}{2}$, B = –2, C = $\frac{3}{2}$

Putting the values of A, B and C in eq. (i),

$\frac{x}{(x-1)(x-2)(x-3)}$

= $\frac{1/2}{x-1}+\frac{-2}{x-2}+\frac{3/2}{x-3}$

$\Rightarrow \int\frac{x}{(x-1)(x-2)(x-3)}\,dx$

= $\frac{1}{2}\int\frac{1}{x-1}\,dx-2\int\frac{1}{x-2}\,dx+\frac{3}{2}\int\frac{1}{x-3}\,dx$

$\Rightarrow \int\frac{x}{(x-1)(x-2)(x-3)}\,dx$

= $\frac{1}{2}\log|x-1|-2\log|x-2|+\frac{3}{2}\log|x-3|+c$

5. $\frac{2x}{x^2+3x+2}$

Ans. $\frac{2x}{x^2+3x+2}$

= $\frac{2x}{x^2+2x+x+2}$

= $\frac{2x}{x(x+2)+1(x+2)}$

= $\frac{2x}{(x+1)(x+2)}$

= $\frac{A}{x+1}+\frac{B}{x+2}$ ....(i)

$\Rightarrow 2x = A(x+2)+B(x+1)$

$\Rightarrow 2x = Ax+2A+Bx+B$

Comparing coefficients of x on both sides A + B = 2 ....(ii)

Comparing constants 2A + B = 0 .....(iii)

Solving eq. (ii) and (iii), we get A = -2 and B = 4

Putting these values of A and B in eq. (i),

$$\frac{2x}{(x+1)(x+2)}=\frac{-2}{x+1}+\frac{4}{x+2}$$

$$\therefore \int\frac{2x}{(x+1)(x+2)}\,dx=-2\int\frac{1}{x+1}\,dx+4\int\frac{1}{x+2}\,dx$$

$= -2log|x+1| + 4log|x+2| + c$

$= 4log|x+2| - 2log|x+1| + c$

6. $\dfrac{1-x^2}{x(1-2x)}$

Ans. $\dfrac{1-x^2}{x(1-2x)}$

$= \dfrac{1-x^2}{x-2x^2}$

$= \dfrac{-x^2+1}{-2x^2+x}$

$= \dfrac{1}{2}+\dfrac{\left(-\dfrac{x}{2}+1\right)}{x(1-2x)}$ (Dividing numerator by denominator)

$$\therefore \int \frac{1-x^2}{x(1-2x)}\, dx = \int \left\{\frac{1}{2}+\frac{\left(-\dfrac{x}{2}+1\right)}{x(1-2x)}\right\} dx$$

$= \dfrac{1}{2}\int 1\, dx + \int \dfrac{-\dfrac{x}{2}+1}{x(1-2x)}\, dx$ ....(i)

Now $\displaystyle\int \frac{-\dfrac{x}{2}+1}{x(1-2x)}\, dx$

$= \dfrac{A}{x}+\dfrac{B}{1-2x}$ ....(ii)

$\Rightarrow -\dfrac{x}{2}+1 = A\,(1-2x)+Bx$

$$\Rightarrow -\frac{x}{2}+1=A-2Ax+Bx$$

Comparing coefficients of x on both sides –2A + B = $\frac{-1}{2}$ .....(iii)

Comparing constants A = 1 ....(iv)

Solving eq. (ii) and (iii), we get A = 1 and B = $\frac{3}{2}$

Putting these values of A and B in eq. (ii),

$$\frac{-\frac{x}{2}+1}{x(1-2x)}=\frac{1}{x}+\frac{3/2}{1-2x}$$

$$\therefore \int\frac{-\frac{x}{2}+1}{x(1-2x)}\,dx=\int\frac{1}{x}\,dx+\frac{3}{2}\int\frac{1}{1-2x}\,dx$$

$$= \log|x|+\frac{3}{2}\log\frac{|1-2x|}{-2\to\text{Coeff. of } x}+c$$

$$= \log|x|-\frac{3}{4}\log|1-2x|+c$$

Putting this value in eq. (i),

$$\int\frac{1-x^2}{x(1-2x)}\,dx$$

$$= \frac{1}{2}x+\log|x|-\frac{3}{4}\log|1-2x|+c$$

Integrate the following function in Exercises 7 to 12.

7. $\frac{x}{(x^2+1)(x-1)}$

Ans. $\frac{x}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$ ....(i)

$\Rightarrow x = (Ax+B)(x-1) + C(x^2+1)$

$\Rightarrow x = Ax^2 - Ax + Bx - B + Cx^2 + C$

Comparing coefficients of x^2, A + C = 0 ....(ii)

Comparing coefficients of x , –A + B = 1 ....(iii)

Comparing constant terms, –B + C = 0 ....(iv)

Solving eq. (ii), (iii) and (iv), we get A = $\frac{-1}{2}$, B = $\frac{1}{2}$ and C = $\frac{1}{2}$

Putting the values of A, B and C in eq. (i), $\frac{x}{(x^2+1)(x-1)} = \frac{\frac{-1}{2}x + \frac{1}{2}}{x^2+1} + \frac{\frac{1}{2}}{x-1}$

$\Rightarrow \frac{x}{(x^2+1)(x-1)} = \frac{-1}{2}.\frac{x}{x^2+1} + \frac{1}{2}.\frac{1}{x^2+1} + \frac{1}{2}.\frac{1}{x-1}$

= $\frac{-1}{4}.\frac{2x}{x^2+1} + \frac{1}{2}.\frac{1}{x^2+1} + \frac{1}{2}.\frac{1}{x-1}$

$\Rightarrow \int \frac{x}{(x^2+1)(x-1)}\,dx = \frac{-1}{4}\int \frac{2x}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx$

$\Rightarrow \int \frac{x}{(x^2+1)(x-1)}\,dx = \frac{-1}{4}\log\left|x^2+1\right| + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log\left|x-1\right| + c$

$$\Rightarrow \int \frac{x}{(x^2+1)(x-1)}\ dx = \frac{-1}{4}\log(x^2+1) + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1| + c$$

$$\Rightarrow \int \frac{x}{(x^2+1)(x-1)}\ dx = \frac{1}{2}\log|x-1| - \frac{1}{4}\log(x^2+1) + \frac{1}{2}\tan^{-1}x + c$$

8. $\frac{x}{(x-1)^2(x+2)}$

Ans. $\frac{x}{(x-1)^2(x+2)}$

$= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$ ....(i)

$$\Rightarrow x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

$$\Rightarrow x = A(x^2+x-2) + B(x+2) + C(x^2-2x+1)$$

$$\Rightarrow x = Ax^2 + Ax - 2A + Bx + 2B + Cx^2 - 2Cx + C$$

Comparing coefficients of x^2 : A + C = 0 ....(ii)

Comparing coefficients of x : A + B – 2C= 1 ....(iii)

Comparing constants: –2A + 2B + C = 0 ....(iv)

On solving eq. (i), (ii) and (iii), we get

A = $\frac{2}{9}$, B = $\frac{1}{3}$, C = $\frac{-2}{9}$

Putting the values of A, B and C in eq. (i), $\frac{x}{(x-1)^2(x+2)}$

$= \frac{\frac{2}{9}}{x-1} + \frac{\frac{1}{3}}{(x-1)^2} + \frac{\frac{-2}{9}}{x+2}$

$\Rightarrow \int \frac{x}{(x-1)^2(x+2)}\, dx$

$= \frac{2}{9}\int \frac{1}{x-1}\, dx + \frac{1}{3}\int \frac{1}{(x-1)^2}\, dx - \frac{2}{9}\int \frac{1}{x+2}\, dx$

$= \frac{2}{9}\log|x-1| + \frac{1}{3}\int (x-1)^{-2}\, dx - \frac{2}{9}\log|x+2| + c$

$= \frac{2}{9}\log|x-1| + \frac{1}{3}\frac{(x-1)^{-1}}{(-1)(1)} - \frac{2}{9}\log|x+2| + c$

$= \frac{2}{9}\left(\log|x-1| - \log|x+2|\right) - \frac{1}{3(x-1)} + c$

$= \frac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + c$

9. $\frac{3x+5}{x^3-x^2-x+1}$

Ans. $\frac{3x+5}{x^3-x^2-x+1}$

$= \frac{3x+5}{x^2(x-1)-1(x-1)}$

$= \frac{3x+5}{(x-1)(x^2-1)}$

= $\frac{3x+5}{(x-1)(x-1)(x+1)}$

= $\frac{3x+5}{(x-1)^2(x+1)}$

= $\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}$ ....(i)

$\Rightarrow 3x+5=A(x-1)(x+1)+B(x+1)+C(x-1)^2$

$\Rightarrow 3x+5=A(x^2-1)+B(x+1)+C(x^2-2x+1)$

$\Rightarrow 3x+5=Ax^2-A+Bx+B+Cx^2-2Cx+C$

Comparing coefficients of x^2 : A + C = 0 ....(ii)

Comparing coefficients of x : B – 2C= 3 ....(iii)

Comparing constants: –2A + B + C = 5 ....(iv)

On solving eq. (i), (ii) and (iii), we get A = $\frac{-1}{2}$, B = 4, C = $\frac{1}{2}$

Putting the values of A, B and C in eq. (i),

$\frac{3x+5}{x^3-x^2-x+1}$

= $\frac{\frac{-1}{2}}{x-1}+\frac{4}{(x-1)^2}+\frac{\frac{1}{2}}{x+1}$

= $\frac{-1}{2}\int\frac{1}{x-1}\,dx+4\int(x-1)^{-2}\,dx+\frac{1}{2}\int\frac{1}{x+1}\,dx$

= $\frac{-1}{2}\log|x-1|+4\frac{(x-1)^{-1}}{(-1)(1)}+\frac{1}{2}\log|x+1|+c$

= $\frac{1}{2}\left(\log|x+1|-\log|x-1|\right)-\frac{4}{(x-1)}+c$

= $\frac{1}{2}\log\left|\frac{x+1}{x-1}\right|-\frac{4}{(x-1)}+c$

10. $\frac{2x-3}{(x^2-1)(2x+3)}$

Ans. $\frac{2x-3}{(x^2-1)(2x+3)}$

= $\frac{2x-3}{(x-1)(x+1)(2x+3)}$

= $\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{2x+3}$ ....(i)

$\Rightarrow 2x-3=A(x+1)(2x+3)+B(x-1)(2x+3)+C(x-1)(x+1)$

$\Rightarrow 2x-3=A(2x^2+5x+3)+B(2x^2+x-3)+C(x^2-1)$

$\Rightarrow 2x-3=2Ax^2+5Ax+3A+2Bx^2+Bx-3B+Cx^2-C$

Comparing coefficients of x^2 : 2A + 2B + C = 0 ....(ii)

Comparing coefficients of x : 5A + B = 2 ....(iii)

Comparing constants: 3A – 3B – C = –3 ....(iv)

On solving eq. (i), (ii) and (iii), we get A = $\frac{-1}{10}$, B = $\frac{5}{2}$, C = $\frac{-24}{5}$

Putting the values of A, B and C in eq. (i),

$$\frac{2x-3}{\left(x^2-1\right)\left(2x+3\right)}$$

= $\frac{\frac{-1}{10}}{x-1}+\frac{\frac{5}{2}}{x+1}+\frac{\frac{-24}{5}}{2x+3}$

$\Rightarrow \int \frac{2x-3}{\left(x^2-1\right)\left(2x+3\right)}\,dx$ = $\frac{-1}{10}\int \frac{1}{x-1}\,dx+\frac{5}{2}\int \frac{1}{x+1}\,dx-\frac{24}{5}\int \frac{1}{2x+3}\,dx$

= $\frac{-1}{10}\log|x-1|+\frac{5}{2}\log|x+1|-\frac{24}{5}\frac{\log|2x+3|}{2\rightarrow \text{Coeff. of } x}+c$

= $\frac{-1}{10}\log|x-1|+\frac{5}{2}\log|x+1|-\frac{12}{5}\log|2x+3|+c$

$=\frac{5}{2}log|x+1|-\frac{1}{10}log|x-1|-\frac{12}{5}log|2x+3|+c$

11. $\frac{5x}{(x+1)\left(x^2-4\right)}$

Ans. $\frac{5x}{(x+1)\left(x^2-4\right)}$

= $\frac{5x}{(x+1)(x+2)(x-2)}$

= $\frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{x-2}$ ....(i)

$\Rightarrow 5x=A(x+2)(x-2)+B(x+1)(x-2)+C(x+1)(x+2)$

$\Rightarrow x=A\left(x^2-4\right)+B\left(x^2-x-2\right)+C\left(x^2+3x+2\right)$

$\Rightarrow x=Ax^2-4A+Bx^2-Bx-2B+Cx^2+3Cx+2C$

Comparing coefficients of x^2 : A + +B + C = 0 ....(ii)

Comparing coefficients of x : –B + 3C= 5 ....(iii)

Comparing constants: –4A – 2B + 2C = 0 ....(iv)

On solving eq. (i), (ii) and (iii), we get A = $\frac{5}{3}$, B = $\frac{-5}{2}$, C = $\frac{5}{6}$

Putting the values of A, B and C in eq. (i),

$$\frac{5x}{(x+1)(x^2-4)}$$

= $\frac{\frac{5}{3}}{x+1}+\frac{\frac{-5}{2}}{(x+2)}+\frac{\frac{5}{6}}{x-2}$

= $\frac{5}{3}\int\frac{1}{x+1}\,dx-\frac{5}{2}\int\frac{1}{(x+2)}\,dx+\frac{5}{6}\int\frac{1}{x-2}\,dx$

$= \frac{5}{3}log|x+1| - \frac{5}{2}log|x+2| + \frac{5}{6}log|x-2| + c$

12. $\frac{x^3+x+1}{x^2-1}$

Ans. $\frac{x^3+x+1}{x^2-1}$

= $x+\frac{2x+1}{x^2-1}$ ....(i)

[On dividing numerator by denominator]

Let $\frac{2x+1}{x^2-1}$

= $\dfrac{2x+1}{(x+1)(x-1)}$

= $\dfrac{A}{x+1}+\dfrac{B}{x-1}$ ....(ii)

$\Rightarrow 2x+1=A(x-1)+B(x+1)$

$\Rightarrow 2x+1=Ax-A+Bx+B$

Comparing coefficients of x : A + B = 2 ....(iii)

Comparing constants: –A + B = 1 ....(iv)

On solving eq. (iii) and (iv), we get A = $\dfrac{1}{2}$, B = $\dfrac{3}{2}$

Putting the values of A, B and C in eq. (ii), $\dfrac{2x+1}{x^2-1}$

= $\dfrac{1/2}{x+1}+\dfrac{3/2}{x-1}$

Putting this value in eq. (i),

$\dfrac{x^3+x+1}{x^2-1}$

= $x+\dfrac{1/2}{x+1}+\dfrac{3/2}{x-1}$

$\Rightarrow \int \dfrac{x^3+x+1}{x^2-1}\,dx$

= $\int x\,dx+\dfrac{1}{2}\int\dfrac{1}{x+1}\,dx+\dfrac{3}{2}\int\dfrac{1}{x-1}\,dx$

= $\frac{x^2}{2}+\frac{1}{2}\log|x+1|+\frac{3}{2}\log|x-1|+c$

Integrate the following function in Exercises 13 to 17.

13. $\frac{2}{(1-x)(1+x^2)}$

Ans. $\frac{2}{(1-x)(1+x^2)}$

= $\frac{A}{1-x}+\frac{Bx+C}{1+x^2}$ ....(i)

$\Rightarrow 2$ = $A(1+x^2)+(Bx+C)(1-x)$

$\Rightarrow 2$ = $A+Ax^2+Bx-Bx^2+C-Cx$

Comparing the coefficients of x^2 A – B = 0 ....(ii)

Comparing the coefficients of x B – C = 0 ....(iii)

Comparing constants A + C = 2 ....(iv)

On solving eq. (ii), (iii) and (iv), we get A = 1, B = 1, C = 1

Putting these values of A, B and C in eq. (i),

$\frac{2}{(1-x)(1+x^2)}$ = $\frac{1}{1-x}+\frac{x+1}{1+x^2}$

= $\frac{1}{1-x}+\frac{x}{1+x^2}+\frac{1}{1+x^2}$ = $\frac{1}{1-x}+\frac{1}{2}.\frac{2x}{1+x^2}+\frac{1}{1+x^2}$

= $\int\frac{1}{1-x}\,dx+\frac{1}{2}\int\frac{2x}{1+x^2}\,dx+\int\frac{1}{1+x^2}\,dx$

= $\dfrac{\log|1-x|}{-1 \to \text{Coeff. of } x} + \dfrac{1}{2}\log\left|1+x^2\right| + \tan^{-1} x + c$

= $-\log|1-x| + \dfrac{1}{2}\log\left(1+x^2\right) + \tan^{-1} x + c$

14. $\dfrac{3x-1}{(x+2)^2}$

Ans. Let I = $\int \dfrac{3x-1}{(x+2)^2}\, dx$ ....(i)

Putting $x+2=t$

$\Rightarrow\ x=t-2$

$\Rightarrow\ \dfrac{dx}{dt}=1$

$\Rightarrow\ dx=dt$

Putting this value in eq. (i),

I = $\int \dfrac{3(t-2)-1}{(t)^2}\, dt$

= $\int \dfrac{3t-6-1}{t^2}\, dt$

= $\int \dfrac{3t-7}{t^2}\, dt$

= $\int \left(\dfrac{3t}{t^2}-\dfrac{7}{t^2}\right) dt$

= $\int \left(\dfrac{3}{t}-\dfrac{7}{t^2}\right) dt$

= $3\int \frac{1}{t}\ dt - 7\int t^{-2}\ dt$

= $3\log|t| - 7\frac{t^{-1}}{-1} + c$

= $3\log|t| + \frac{7}{t} + c$

= $3\log|x+2| + \frac{7}{x+2} + c$

15. $\frac{1}{x^4-1}$

Ans. $\frac{1}{x^4-1}$

= $\frac{1}{(x^2-1)(x^2+1)}$

Putting $x^2 = y$, $\frac{1}{x^4-1}$

= $\frac{1}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1}$ .....(i)

$\Rightarrow 1 = A(y+1) + B(y-1)$

$\Rightarrow 1 = Ay + A + By - B$

Comparing the coefficients of y A + B = 0 ......(ii)

Comparing constants A – B = 1 ....(iii)

On solving the eq. (ii) and (iii), we get A = $\frac{1}{2}$, B = $\frac{-1}{2}$

Putting the values of A, B and y in eq. (i),

$$\frac{1}{x^4-1}=\frac{\frac{1}{2}}{x^2-1}+\frac{\frac{-1}{2}}{x^2+1}$$

$$\Rightarrow \int \frac{1}{x^4-1}\ dx=\frac{1}{2}\int \frac{1}{x^2-1}\ dx-\frac{1}{2}\int \frac{1}{x^2+1}\ dx$$

= $\frac{1}{2}.\frac{1}{2.1}\log\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\tan^{-1}x+c$

= $\frac{1}{4}\log\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\tan^{-1}x+c$

16. $\frac{1}{x\left(x^n+1\right)}$

Ans. Let I = $\int \frac{1}{x\left(x^n+1\right)}\ dx$

Multiplying both numerator and denominator by nx^{n-1},

$$\left[\because \frac{d}{dx}\left(x^n+1\right)=nx^{n-1}\right]$$

I = $\int \frac{nx^{n-1}}{nx^{n-1}.x\left(x^n+1\right)}\ dx$

= $\frac{1}{n}\int \frac{nx^{n-1}}{x^n\left(x^n+1\right)}\ dx$ ....(i)

Putting $x^n=t$

$$\Rightarrow nx^{n-1}=\frac{dt}{dx}$$

$$\Rightarrow nx^{n-1}\ dx=dt$$

$\therefore$ From eq. (i),

I = $\frac{1}{n}\int \frac{dt}{t(t+1)}$

= $\frac{1}{n}\int \frac{1}{t(t+1)}\ dt$

= $\frac{1}{n}\int \frac{t+1-t}{t(t+1)}\ dt$

= $\frac{1}{n}\int \frac{t+1}{t(t+1)}-\frac{t}{t(t+1)}\ dt$

= $\frac{1}{n}\left[\int \frac{1}{t}\ dt+\int \frac{1}{t+1}\ dt\right]$

= $\frac{1}{n}\left[\log|t|-\log|t+1|\right]+c$

= $\frac{1}{n}\left[\log\left|x^n\right|-\log\left|x^n+1\right|\right]+c$

= $\frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right|+c$

17. $\frac{\cos x}{(1-\sin x)(2-\sin x)}$

Ans. Let I = $\int \frac{\cos x}{(1-\sin x)(2-\sin x)}\ dx$ ....(i)

Putting $\sin x=t$

$\Rightarrow\ \cos x=\frac{dt}{dx}$

$\Rightarrow \cos x \, dx = dt$

$\therefore$ From eq. (i), I = $\int \frac{1}{(1-t)(2-t)} \, dt$

= $\int \frac{(2-t)-(1-t)}{(1-t)(2-t)} \, dt$

= $\int \left(\frac{(2-t)}{(1-t)(2-t)} - \frac{(1-t)}{(1-t)(2-t)} \right) dt$

= $\int \left(\frac{1}{(1-t)} - \frac{1}{(2-t)} \right) dt$

= $\int \frac{1}{1-t} \, dt - \int \frac{1}{2-t} \, dt$

= $\frac{\log|1-t|}{-1 \rightarrow \text{Coeff. of } t} - \frac{\log|2-t|}{-1} + c$

= $-\log|1-t| + \log|2-t| + c$

= $\log\left|\frac{2-t}{1-t}\right| + c$

= $\log\left|\frac{2-\sin x}{1-\sin x}\right| + c$

Integrate the following function in Exercises 18 to 21.

18. $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$

Ans. $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$ ....(i)

Putting $x^2 = y$,

$$\frac{(y+1)(y+2)}{(y+3)(y+4)}$$

= $\frac{y^2+3y+2}{y^2+7y+12}$ ....(ii)

Dividing numerator by denominator,

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$$

= $1+\frac{(-4y-10)}{(y+3)(y+4)}$ ....(iii)

Let $\frac{(-4y-10)}{(y+3)(y+4)} = \frac{A}{y+3} + \frac{B}{y+4}$ ....(iv)

$$\Rightarrow -4y-10 = A(y+4)+B(y+3)$$

$$\Rightarrow -4y-10 = Ay+4A+By+3B$$

Comparing coefficients of y A + B = –4 ....(v)

Comparing constants 4A + 3B = –10 ....(vi)

On solving eq. (v) and (vi), we get A = 2, B = –6

Putting the values of A, B and y in eq. (iii),

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$$

= $1+\frac{2}{x^2+3}-\frac{6}{x^2+4}$

$$\Rightarrow \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}\,dx$$

$$= \int \left(1+\frac{2}{x^2+3}-\frac{6}{x^2+4}\right)dx$$

$$= \int 1\,dx + 2\int \frac{1}{x^2+\left(\sqrt{3}\right)^2}\,dx - 6\int \frac{1}{x^2+2^2}\,dx$$

$$= x + 2.\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 6.\frac{1}{2}\tan^{-1}\frac{x}{2} + c$$

$$= x + \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 3\tan^{-1}\frac{x}{2} + c$$

19. $\frac{2x}{(x^2+1)(x^2+3)}$

Ans. Let I = $\int \frac{2x}{(x^2+1)(x^2+3)}\,dx$ ....(i)

Putting $x^2 = t \Rightarrow 2x\,dx = dt$

$\therefore$ From eq. (i),

I = $\int \frac{dt}{(t+1)(t+3)}$

$$= \frac{1}{2}\int \frac{2}{(t+1)(t+3)}\,dt$$

$$= \frac{1}{2}\int \frac{(t+3)-(t+1)}{(t+1)(t+3)}\,dt$$

= $\frac{1}{2}\int\left(\frac{(t+3)}{(t+1)(t+3)}-\frac{(t+1)}{(t+1)(t+3)}\right)dt$

= $\frac{1}{2}\int\left(\frac{1}{(t+1)}-\frac{1}{(t+3)}\right)dt$

= $\frac{1}{2}\left[\log|t+1|-\log|t+3|\right]+c$

= $\frac{1}{2}\left[\log\left|\frac{t+1}{t+3}\right|\right]+c$

= $\frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right|+c$

= $\frac{1}{2}\log\left(\frac{x^2+1}{x^2+3}\right)+c$

20. $\frac{1}{x(x^4-1)}$

Ans. Let I = $\int\frac{1}{x(x^4-1)}dx$

= $\int\frac{4x^3}{4x^3.x(x^4-1)}dx$

= $\frac{1}{4}\int\frac{4x^3}{x^4(x^4-1)}dx$ ...(i)

$\left[\because \frac{d}{dx}(x^4-1)=4x^3\right]$

Putting $x^4=t$

$\Rightarrow 4x^3 = \frac{dt}{dx}$

$\Rightarrow 4x^3\ dx = dt$

Putting this value in eq. (i),

I = $\frac{1}{4}\int \frac{dt}{t(t-1)}$

= $\frac{1}{4}\int \frac{t-(t-1)}{t(t-1)}\ dt$

I = $\frac{1}{4}\int \left(\frac{t}{t(t-1)} - \frac{(t-1)}{t(t-1)}\right)\ dt$

= $\frac{1}{4}\int \left(\frac{1}{(t-1)} - \frac{1}{t}\right)\ dt$

= $\frac{1}{4}\left[\int \frac{1}{t-1}\ dt - \int \frac{1}{t}\ dt\right]$

= $\frac{1}{4}\left[\log|t-1| - \log|t|\right] + c$

= $\frac{1}{4}\log\left|\frac{t-1}{t}\right| + c$

= $\frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right| + c$

21. $\frac{1}{(e^x-1)}$

Ans. Let I = $\int \frac{1}{e^x-1}\ dx$ ....(i)

Putting $e^x = t$

$$\Rightarrow e^x = \frac{dt}{dx}$$

$$\Rightarrow e^x \ dx = dt$$

$$\Rightarrow dx = \frac{dt}{e^x}$$

$\therefore$ From eq. (i),

$$I = \int \frac{1}{t-1} \frac{dt}{e^x}$$

$$= \int \frac{1}{t-1} \frac{dt}{t}$$

$$= \int \frac{1}{t(t-1)} \ dt$$

$$= \int \frac{t-(t-1)}{t(t-1)} \ dt$$

$$= \int \left(\frac{t}{t(t-1)} - \frac{(t-1)}{t(t-1)} \right) dt$$

$$= \int \left(\frac{1}{(t-1)} - \frac{1}{t} \right) dt$$

$$= \int \frac{1}{t-1} \ dt - \int \frac{1}{t} \ dt$$

$$= \log |t-1| - \log |t| + c$$

$$= \log \left| \frac{t-1}{t} \right| + c$$

$= \log\left|\frac{e^x-1}{e^x}\right|+c$

Choose the correct answer in each of the Exercise 22 and 23.

22. $\int \frac{x\,dx}{(x-1)(x-2)}$ **equals:**

(A) $\log\left|\frac{(x-1)^2}{x-2}\right|+C$

(B) $\log\left|\frac{(x-2)^2}{x-1}\right|+C$

(C) $\log\left|\left(\frac{x-1}{x-2}\right)^2\right|+C$

(D) $\log\left|(x-1)(x-2)\right|+C$

Ans. Let $\frac{x}{(x-1)(x-2)}=\frac{A}{x-1}+\frac{B}{x-2}$ ....(i)

$\Rightarrow\ x = A(x-2)+B(x-1)$

$\Rightarrow\ x = Ax-2A+Bx-B$

Comparing coefficients of x A + B = 1 ....(ii)

Comparing constants –2A – B = 0 ....(iii)

On solving eq. (ii) and (iii), we get A = –1, B = 2

Putting these values of A and B in eq. (i),

$$\frac{x}{(x-1)(x-2)}=\frac{-1}{x-1}+\frac{2}{x-2}$$

$\Rightarrow \int \frac{x}{(x-1)(x-2)}\,dx = -\int \frac{1}{x-1}\,dx + 2\int \frac{1}{x-2}\,dx$

= $-1\log|x-1| + 2\log|x-2| + c$

= $\log\left|(x-2)^2\right| - \log|x-1| + c$

= $\log\left|\frac{(x-2)^2}{x-1}\right| + c$

Therefore, option (B) is correct.

23. $\int \frac{dx}{x(x^2+1)}$ **equals:**

(A) $\log|x| - \frac{1}{2}\log(x^2+1) + C$

(B) $\log|x| + \frac{1}{2}\log(x^2+1) + C$

(C) $-\log|x| - \frac{1}{2}\log(x^2+1) + C$

(D) $\frac{1}{2}\log|x| + \log(x^2+1) + C$

Ans. Let I = $\int \frac{1}{x(x^2+1)}\,dx$

= $\int \frac{2x}{2x.x(x^2+1)}\,dx$

= $\int \frac{2x}{2x^2(x^2+1)}\,dx$ ...(i)

$$\left[\because \frac{d}{dx}\left(x^2+1\right)=2x\right]$$

Putting $x^2 = t$

$$\Rightarrow\ 2x=\frac{dt}{dx}$$

$$\Rightarrow\ 2x\ dx=dt$$

Putting this value in eq. (i),

I = $\displaystyle\int \frac{dt}{2t\left(t+1\right)}$

= $\displaystyle\frac{1}{2}\int \frac{\left(t+1\right)-t}{t\left(t+1\right)}\ dt$

I = $\displaystyle\frac{1}{2}\int\left(\frac{t+1}{t\left(t+1\right)}-\frac{t}{t\left(t+1\right)}\right)\ dt$

= $\displaystyle\frac{1}{2}\int\left(\frac{1}{t}-\frac{1}{t+1}\right)\ dt$

= $\displaystyle\frac{1}{2}\left[\int\frac{1}{t}\ dt-\int\frac{1}{t+1}\ dt\right]$

= $\displaystyle\frac{1}{2}\left[\log\left|t\right|-\log\left|t+1\right|\right]+c$

= $\displaystyle\frac{1}{2}\left[2\log\left|x^2\right|-\log\left(x^2+1\right)\right]+c$

= $\displaystyle\log\left|x\right|-\frac{1}{2}\log\left(x^2+1\right)+c$

Therefore, option (A) is correct.