CBSE Class–12 Mathematics

NCERT solution

Chapter - 4

Determinants - Miscellaneous Exercise

1. Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ **is independent of** θ.

Ans. Let $\Delta = \begin{bmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{bmatrix}$

Expanding along first row,

$$\Delta = x\begin{vmatrix} -x & 1 \\ 1 & x \end{vmatrix} - \sin\theta\begin{vmatrix} -\sin\theta & 1 \\ \cos\theta & x \end{vmatrix} + \cos\theta\begin{vmatrix} -\sin\theta & -x \\ \cos\theta & 1 \end{vmatrix}$$

$$\Rightarrow \Delta = x\left(-x^2-1\right) - \sin\theta\left(-x\sin\theta - \cos\theta\right) + \cos\theta\left(-\sin\theta + x\cos\theta\right)$$

$$\Rightarrow \Delta = -x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta$$

$\Rightarrow \Delta = -x^3 - x + x\left(\sin^2\theta + \cos^2\theta\right)$ = $-x^3 - x + x$ = $-x^3$ which is independent of θ.

2. Without expanding the determinants, prove that: $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$.

Ans. L.H.S. = $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$

Multiplying R_1 by a, R_2 by b and R_3 by c.

$$= \begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix}$$

$$= \frac{abc}{abc}\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$$

[Interchanging C_1 and C_3]

$$= -\begin{vmatrix} 1 & a^3 & a^2 \\ 1 & b^3 & b^2 \\ 1 & c^3 & c^2 \end{vmatrix}$$

[Interchanging C_2 and C_3]

$$= (-)(-)\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \text{RHS}$$

Proved.

3. Evaluate: $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Ans. Let $\Delta = \begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Expanding along first row,

=

$\cos\alpha\cos\beta(\cos\alpha\cos\beta-0)-\cos\alpha\sin\beta(-\cos\alpha\sin\beta-0)-\sin\alpha\left(-\sin\alpha\sin^2\beta-\sin\alpha\cos^2\beta\right)$

= $\cos^2\alpha\cos^2\beta+\cos^2\alpha\sin^2\beta+\sin^2\alpha\left(\sin^2\beta+\cos^2\beta\right)$

= $\cos^2\alpha\left(\cos^2\beta+\sin^2\beta\right)+\sin^2\alpha\left(\sin^2\beta+\cos^2\beta\right)$

= $\cos^2\alpha+\sin^2\alpha$

= 1

4. If a, b and c are real numbers and $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$. Show that either a+b+c=0 or a=b=c

Ans. Given: $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$

$\left[R_1 \to R_1+R_2+R_3\right]$

$$\Rightarrow \Delta = \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$$

$$\Rightarrow \Delta = 2(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$$

Here, Either $2(a+b+c)=0$

$\Rightarrow a+b+c=0$ ....(i)

Or $\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$

$[C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1]$

$$\Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ c+a & a+b-c-a & b+c-c-a \\ a+b & b+c-a-b & c+a-a-b \end{vmatrix} = 0$$

$\Rightarrow \begin{vmatrix} b-c & b-a \\ c-a & c-b \end{vmatrix} = 0$ [Expanding along first row]

$\Rightarrow (b-c)(c-b)-(b-a)(c-a)=0$

$\Rightarrow bc-b^2-c^2+bc-bc+ab+ac-a^2=0$

$\Rightarrow -a^2-b^2-c^2+ab+bc+ca=0$

$\Rightarrow 2a^2+2b^2+2c^2-2ab-2bc-2ca=0$

$\Rightarrow a^2+a^2+b^2+b^2+c^2+c^2-2ab-2bc-2ca=0$

$\Rightarrow\ \left(a^2+b^2-2ab\right)+\left(b^2+c^2-2bc\right)+\left(a^2+c^2-2ca\right)=0$

$\Rightarrow\ (a-b)^2+(b-c)^2+(c-a)^2=0$

$\Rightarrow\ a-b=0$ and $b-c=0$ and $c-a=0$

$\left[x^2+y^2+z^2=0, \text{then } x=0, y=0, z=0\right]$

$\Rightarrow\ a=b$ and $b=c$ and $c=a$ ....(ii)

Therefore, from eq. (i) and (ii),

either $a+b+c=0$ or $a=b=c$

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5. Solve the equation: $\Delta=\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix}=0, a\neq 0$

Ans. Given: $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix}=0$

$\left[R_1\rightarrow R_1+R_2+R_3\right]$

$\Rightarrow\begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x+a & x \\ x & x & x+a \end{vmatrix}=0$

$\Rightarrow\ (3x+a)\begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix}=0$

Either $3x+a=0$

$\Rightarrow x = \frac{-a}{3}$ ....(i)

Or $\begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$

$[C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1]$

$\Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ x & a & 0 \\ x & 0 & a \end{vmatrix} = 0$

$\Rightarrow 1(a^2 - 0) = 0$

$\Rightarrow a^2 = 0$

$\Rightarrow a = 0$

But this is contrary as given that $a \neq 0$.

Therefore, from eq. (i), $x = \frac{-a}{3}$ is only the solution.

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6. Prove that: $\Delta = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

Ans. L.H.S. = $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$

$$= \begin{vmatrix} a^2 & bc & c(a+c) \\ a(a+b) & b^2 & ac \\ ab & b(b+c) & c^2 \end{vmatrix}$$

$$= abc\begin{vmatrix} a & c & (a+c) \\ (a+b) & b & a \\ b & (b+c) & c \end{vmatrix}$$

$[R_1 \to R_1 - R_2 - R_3]$

$$= abc\begin{vmatrix} a-a-b-b & c-b-b-c & a+c-a-c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

$$= abc\begin{vmatrix} -2b & -2b & 0 \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

$[C_2 \to C_2 - C_1]$

$$= abc\begin{vmatrix} -2b & 0 & 0 \\ a+b & -a & a \\ b & c & c \end{vmatrix}$$

$= abc(-2b)(-ac-ac)$

$= 4a^2b^2c^2$ = R.H.S. Proved.

7. If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ **and B =** $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, **find** $(AB)^{-1}$.

Ans. Given: $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Since, $(AB)^{-1} = B^{-1}A^{-1}$ [Reversal law] ……….(i)

Now $|B| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix}$

= $1(3-0)-2(-1-0)+(-2)(2-0)$ = $3+2-4=1\neq 0$

Therefore, B^{-1} exists.

$\therefore$ $B_{11}=3, B_{12}=1, B_{13}=2$ and $B_{21}=2, B_{22}=1, B_{23}=2$ and $B_{31}=6, B_{32}=2, B_{33}=5$

$\therefore$ adj. B = $\begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}'$ = $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

$\therefore$ $B^{-1} = \frac{1}{|B|}(\text{adj. B}) = \frac{1}{1}\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

From eq. (i), $(AB)^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

$$\Rightarrow (AB)^{-1} = \begin{bmatrix} 9-30+30 & -3+12-12 & 3-10+12 \\ 3-15+10 & -1+6-4 & 1-5+4 \\ 6-30+25 & -2+12-10 & 2-10+10 \end{bmatrix}$$

$$= \begin{vmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

8. Let A = $\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$, **verify that:**

(i) $(\text{adj. } A)^{-1} = \text{adj. } (A^{-1})$

(ii) $(A^{-1})^{-1} = A$

Ans. Given: Matrix A = $\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

$$\therefore |A| = \begin{vmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{vmatrix}$$

$$\Rightarrow |A| = 1(15-1)-(-2)(-10-1)+1(-2-3) = 14-22-5 = -13 \neq 0$$

Therefore, A^{-1} exists.

$\therefore A_{11} = 14, A_{12} = 11, A_{13} = -5$ and $A_{21} = 11, A_{22} = 4, A_{23} = -3$

and $A_{31} = -5, A_{32} = -3, A_{33} = -1$

$\therefore$ adj. A = $\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$ = B (say)

$\therefore\ A^{-1} = \frac{1}{|A|}(\text{adj. } A)$ = $\frac{-1}{13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$ ....(i)

$\Rightarrow |B| = \begin{vmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{vmatrix}$

= $14(-4-9)-11(-11-15)-5(-33+20)$ = $169 \neq 0$

Therefore, B^{-1} exists.

$\therefore\ B_{11} = -13, B_{12} = 26, B_{13} = -13$ and $B_{21} = 26, B_{22} = -39, B_{23} = -13$

and $B_{31} = -13, B_{32} = -13, B_{33} = -65$

$\therefore$ adj. B = $\begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix}$ = $-13\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

$\Rightarrow B^{-1} = (\text{adj. } A)^{-1} = \frac{1}{|B|}(\text{adj. } B)$

= $\frac{1}{169}(-13)\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

$= \frac{-1}{13}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ ....(ii)

Now to find $\text{adj. } A^{-1} = \text{adj. } C$ (say), where

$C = A^{-1} = \frac{-1}{13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$

$= \begin{bmatrix} -14/13 & -11/13 & 5/13 \\ -11/13 & -4/11 & 3/13 \\ 5/13 & 3/11 & 1/13 \end{bmatrix}$

$C = A^{-1} = \frac{-14}{13}\left(\frac{-4}{169} - \frac{9}{169}\right) - \left(\frac{-11}{13}\right)\left(\frac{-11}{169} - \frac{15}{169}\right) + \frac{5}{13}\left(\frac{-33}{169} + \frac{20}{169}\right)$

$C = A^{-1} = \frac{-14}{13}\left(\frac{-13}{169}\right) + \frac{11}{13}\left(\frac{-26}{169}\right) + \frac{5}{13}\left(\frac{-13}{169}\right) = \frac{14}{169} - \frac{22}{169} - \frac{5}{169} = \frac{-13}{169} = \frac{-1}{13} \neq 0$

Therefore, C^{-1} exists.

$\therefore\ C_{11} = \frac{-1}{13}, C_{12} = \frac{2}{13}, C_{13} = \frac{-1}{13}$ and $C_{21} = \frac{2}{13}, C_{22} = \frac{-3}{13}, C_{23} = \frac{-1}{13}$

and $C_{31} = \frac{-1}{13}, C_{32} = \frac{-1}{13}, C_{33} = \frac{-5}{13}$

$\therefore$ adj. A = $\text{adj. } \left(A^{-1}\right) = \begin{vmatrix} -1/13 & 2/13 & -1/13 \\ 2/13 & -3/13 & -1/13 \\ -1/13 & -1/13 & -5/13 \end{vmatrix}$

= $\frac{-1}{13}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ ....(iii)

Again $\left(A^{-1}\right)^{-1} = C^{-1} = \frac{1}{|C|}(\text{adj. } C)$

= $\frac{1}{-1/13}\left(\frac{-1}{13}\right)\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

= $\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ = A (given)

(i) $(\text{adj. } A)^{-1} = \text{adj. }\left(A^{-1}\right)$

$\Rightarrow \frac{-1}{13}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = \frac{-1}{13}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

[From eq. (ii) and (iii)]

(ii) $\left(A^{-1}\right)^{-1} = A$

$\Rightarrow \frac{-1}{13}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

9. Evaluate: $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

Ans. Let $\Delta = \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

$[R_1 \to R_1 + R_2 + R_3]$

= $\begin{vmatrix} 2(x+y) & 2(x+y) & 2(x+y) \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

= $2(x+y)\begin{vmatrix} 1 & 1 & 1 \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

$[C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1]$

= $2(x+y)\begin{vmatrix} 1 & 0 & 0 \\ y & x+y-y & x-y \\ x+y & x-x-y & y-x-y \end{vmatrix}$

= $2(x+y)\begin{vmatrix} 1 & 0 & 0 \\ y & x & x-y \\ x+y & -y & -x \end{vmatrix}$

= $2(x+y).1\begin{vmatrix} x & x-y \\ -y & -x \end{vmatrix}$

= $2(x+y)\{-x^2 + y(x-y)\}$

= $2(x+y)(-x^2 + xy - y^2)$

= $-2(x+y)(x^2 - xy + y^2)$

= $-2\left(x^3+y^3\right)$

10. Evaluate: $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

Ans. Let $\Delta = \begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

$\left[\mathrm{R}_2 \to \mathrm{R}_2 - \mathrm{R}_1 \text{ and } \mathrm{R}_3 \to \mathrm{R}_3 - \mathrm{R}_1\right]$

= $\begin{vmatrix} 1 & x & y \\ 0 & x+y-x & 0 \\ 0 & 0 & x+y-y \end{vmatrix}$

= $\begin{vmatrix} 1 & x & y \\ 0 & y & 0 \\ 0 & 0 & x \end{vmatrix}$

= $1\begin{vmatrix} y & 0 \\ 0 & x \end{vmatrix}$

= xy

Using properties of determinants in Exercises 11 to 15, prove that:

11. $\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$

Ans. L.H.S. = $\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix}$

$[C_3 \to C_3 + C_1]$

= $\begin{vmatrix} \alpha & \alpha^2 & \alpha+\beta+\gamma \\ \beta & \beta^2 & \alpha+\beta+\gamma \\ \gamma & \gamma^2 & \alpha+\beta+\gamma \end{vmatrix}$

$=(\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta & \beta^2 & 1 \\ \gamma & \gamma^2 & 1 \end{vmatrix}$

$[R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$

$= (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta-\alpha & \beta^2-\alpha^2 & 0 \\ \gamma-\alpha & \gamma^2-\alpha^2 & 0 \end{vmatrix}$

Expanding along third column,

$(\alpha+\beta+\gamma)\begin{vmatrix} \beta-\alpha & \beta^2-\alpha^2 \\ \gamma-\alpha & \gamma^2-\alpha^2 \end{vmatrix}$

$= (\alpha+\beta+\gamma)\begin{vmatrix} \beta-\alpha & (\beta-\alpha)(\beta+\alpha) \\ \gamma-\alpha & (\gamma-\alpha)(\gamma+\alpha) \end{vmatrix}$

$= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} 1 & (\beta+\alpha) \\ 1 & (\gamma+\alpha) \end{vmatrix}$

= $(\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)(\gamma+\alpha-\beta-\alpha)$

= $(\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)(\gamma-\beta)$

= $(\alpha+\beta+\gamma)[-(\alpha-\beta)](\gamma-\alpha)[-(\beta-\gamma)]$

= $(\alpha+\beta+\gamma)(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$ = R.H.S.

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12. $\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x)$

Ans. L.H.S. = $\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$

= $\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix}$

= $\Delta_1 + \Delta_2$ (say) ....(i)

Now $\Delta_2 = \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix}$

= $pxyz\Delta_2 = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

= $-pxyz\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix}$

= $pxyz\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$

$\Rightarrow$ From eq. (i), L.H.S. = $\Delta_1 + pxyz\Delta_1$ ....(ii)

Now $\Delta_1 = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$

$[R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$

= $\begin{vmatrix} x & x^2 & 1 \\ y-x & y^2-x^2 & 0 \\ z-x & z^2-x^2 & 1 \end{vmatrix}$

Expanding along third column

, $\Delta_1 = \begin{vmatrix} y-x & y^2-x^2 \\ z-x & z^2-x^2 \end{vmatrix}$

= $\begin{vmatrix} y-x & (y-x)(y+x) \\ z-x & (z-x)(z+x) \end{vmatrix}$

= $(y-x)(z-x)\begin{vmatrix} 1 & y+x \\ 1 & z+x \end{vmatrix}$

= $(y-x)(z-x)(z+x-y-x)$

= $(y-x)(z-x)(z-y)$

= $(x-y)(y-z)(z-x)$

$\therefore$ From eq. (i), L.H.S.

= $(y-x)(z-x)(z-y)+pxyz(y-x)(z-x)(z-y)$

= $(1+pxyz)(y-x)(z-x)(z-y)$ = R.H.S.

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13. $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$

Ans. L.H.S. = $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$

$[C_1 \to C_1 + C_2 + C_3]$

= $\begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$

= $(a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$

$[R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$

= $(a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & -b+a \\ 0 & -c+a & 2c+a \end{vmatrix}$

$= (a+b+c).1\begin{vmatrix} 2b+a & a-b \\ a-c & 2c+a \end{vmatrix}$

$= (a+b+c)\left[(2b+a)(2c+a)-(a-b)(a-c)\right]$

$(a+b+c)[4bc+2ab+2ac+a^2-a^2+ac+ab-bc]$

$= (a+b+c)(3ab+3bc+3ac)$

$= 3(a+b+c)(ab+bc+ac)$ = R.H.S.

14. $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$

Ans. L.H.S. = $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$

$[R_2 \to R_2 - 2R_1 \text{ and } R_3 \to R_3 - 3R_1]$

$= \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 3 & 7+3p \end{vmatrix}$

$= 1\begin{vmatrix} 1 & 2+p \\ 3 & 7+3p \end{vmatrix} - 0 + 0$

$= 7+3p-3(2+p)$

$= 7+3p-6-3p$

=> 1 = R.H.S.

15. $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$ **= 0**

Ans. L.H.S. = $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$

= $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta-\sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta-\sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta-\sin\gamma\sin\delta \end{vmatrix}$

$\left[C_3 \to C_3+(\sin\delta)C_1\right]$

= $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta-\sin\alpha\sin\delta+\sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta-\sin\beta\sin\delta+\sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta-\sin\gamma\sin\delta+\sin\gamma\sin\delta \end{vmatrix}$

= $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta \end{vmatrix}$

= $\cos\delta\begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha \\ \sin\beta & \cos\beta & \cos\beta \\ \sin\gamma & \cos\gamma & \cos\gamma \end{vmatrix}$

= $\cos\delta(0)$ [$\because$ C_2 and C_3 have become identical]

= 0 = R.H.S.

16. Solve the system of the following equations: (Using matrices):

$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4;\quad \frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1;\quad \frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2$

Ans. Putting $\frac{1}{x}=u, \frac{1}{y}=v$ and $\frac{1}{z}=w$ in the given equations,

$2u+3v+10w=4;\ 4u-6v+5w=1;\ 6u+9v-20w=2$

$\therefore$ the matrix form of given equations is $\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix}=\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$ [AX= B]

Here, A = $\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$, X = $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$ and B = $\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

$\therefore\ |A|=\begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix}$

= $2(120-45)-3(-80-30)+10(36+36)$

= $150+330+750=1200 \neq 0$

$\therefore\ A^{-1}$ exists and unique solution is $X=A^{-1}B$ ....(i)

Now $A_{11}=75, A_{12}=110, A_{13}=72$ and $A_{21}=150, A_{22}=-100, A_{23}=0$

and $A_{31}=75, A_{32}=30, A_{33}=-24$

$\therefore$ adj. A = $\begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}'$ = $\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$

And $A^{-1} = \frac{\text{adj.A}}{|A|} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$

$\therefore$ From eq. (i),

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 300+150+150 \\ 440-100+60 \\ 288+0-48 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ 1/3 \\ 1/5 \end{bmatrix}$$

$$\therefore\ u = \frac{1}{2}, v = \frac{1}{3}, w = \frac{1}{5}$$

$$\Rightarrow x = \frac{1}{u} = 2, y = \frac{1}{v} = 3, z = \frac{1}{w} = 5$$

Choose the correct answer in Exercise 17 to 19.

17. If a, b, c are in A.P., then the determinant $\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$ **is:**

(A) 0

(B) 1

(C) x

(D) $2x$

Ans. According to question, $b-a=c-b$ ....(i)

Let $\Delta = \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$

$[R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_2]$

$= \begin{vmatrix} x+2 & x+3 & x+2a \\ 1 & 1 & 2(b-a) \\ 1 & 1 & 2(c-b) \end{vmatrix}$

$= \begin{vmatrix} x+2 & x+3 & x+2a \\ 1 & 1 & 2(b-a) \\ 1 & 1 & 2(b-a) \end{vmatrix}$

[From eq. (i)] = 0 [$\because$ R_2 and R_3 have become identical]

Therefore, option (A) is correct.

18. If x, y, z **are non-zero real numbers, then the inverse of matrix A =** $\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ **is:**

(A) $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{1} \end{bmatrix}$

(B) $xyz\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{1} \end{bmatrix}$

(C) $\frac{1}{xyz}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$

(D) $\frac{1}{xyz}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Ans. Given: Matrix A = $\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$

$\therefore \ |A| = \begin{vmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{vmatrix}$

$\Rightarrow \ |A| = x(yz - 0) - 0 + 0 = xyz \neq 0$

$\therefore \ A^{-1}$ exists and unique solution is $X = A^{-1}B$ ....(i)

Now $A_{11} = yz, A_{12} = 0, A_{13} = 0$ and $A_{21} = 0, A_{22} = xz, A_{23} = 0$ and $A_{31} = 0, A_{32} = 0, A_{33} = xy$

$\therefore$ adj. A = $\begin{vmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{vmatrix}$ = $\begin{vmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{vmatrix}$

And $A^{-1} = \frac{\text{adj.A}}{|A|} = \frac{1}{xyz}\begin{vmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{vmatrix}$

= $\begin{vmatrix} \frac{yz}{xyz} & 0 & 0 \\ 0 & \frac{xz}{xyz} & 0 \\ 0 & 0 & \frac{xy}{xyz} \end{vmatrix}$

= $\begin{vmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{vmatrix}$

= $\begin{vmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{vmatrix}$

Therefore, option (A) is correct.

19. Let A = $\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, **where** $0 \le \theta \le 2\pi$. **Then:**

(A) Det (A) = 0

(B) Det (A) $\in (2, \infty)$

(C) Det (A) $\in (2, 4)$

(D) Det (A) $\in [2, 4]$

Ans. Given: Matrix A = $\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$

$$\therefore \ |A| = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$$

$$\Rightarrow |A| = 1(1+\sin^2\theta) - \sin\theta(-\sin\theta + \sin\theta) + 1(\sin^2\theta + 1)$$

$$\Rightarrow |A| = 1 + \sin^2\theta + 1 + \sin^2\theta = 2 + 2\sin^2\theta$$ ....(i)

Since $-1 \le \sin\theta \le 1$

$\Rightarrow 0 \le \sin^2\theta \le 1$ [$\because \sin^2\theta$ cannot be negative]

$\Rightarrow 0 \le 2\sin^2\theta \le 2$

$\Rightarrow 2 \le 2 + 2\sin^2\theta \le 4$

$\Rightarrow 2 \le \text{Det. A} \le 4$

Therefore, option (D) is correct.