CBSE Class–12 Mathematics

NCERT solution

Chapter - 6

Application of Derivatives - Exercise 6.5

1. Find the maximum and minimum values, if any, of the following functions given by:

(i) $f(x)=(2x-1)^2+3$

(ii) $f(x)=9x^2+12x+2$

(iii) $f(x)=-(x-1)^2+10$

(iv) $g(x)=x^3+1$

Ans. (i) Given: $f(x)=(2x-1)^2+3$

Since $(2x-1)^2\geq 0$ for all $x\in$ R

Adding 3 both sides, $(2x-1)^2+3\geq 0+3 \Rightarrow f(x)\geq 3$

Therefore, the minimum value of $f(x)$ is 3 when $2x-1=0$, i.e., $x=\frac{1}{2}$

This function does not have a maximum value.

(ii) Given: $f(x)=9x^2+12x+2$

$$\Rightarrow f(x)=9\left(x^2+\frac{4x}{3}+\frac{2}{9}\right)$$

$$\Rightarrow f(x)=9\left(x^2+\frac{4x}{3}+\left(\frac{2}{3}\right)^2-\left(\frac{2}{3}\right)^2+\frac{2}{9}\right)$$

$= 9\left[\left(x+\frac{2}{3}\right)^2 - \frac{4}{9}+\frac{2}{9}\right]$

$\Rightarrow f(x) = 9\left(x+\frac{2}{3}\right)^2 - 2$ ....(i)

Since $9\left(x+\frac{2}{3}\right)^2 \geq 0$ for all $x \in$ R

Subtracting 2 from both sides, $9\left(x+\frac{2}{3}\right)^2 - 2 \geq 0-2$

$\Rightarrow f(x) \geq -2$

Therefore, minimum value of $f(x)$ is -2 and is obtained when $x+\frac{2}{3}=0$, i.e., $x=\frac{-2}{3}$

And this function does not have a maximum value.

(iii) Given: $f(x) = -(x-1)^2+10$ ....(i)

Since $(x-1)^2 \geq 0$ for all $x \in$ R

Multiplying both sides by -1 and adding 10 both sides,

$-(x-1)^2+10 \leq 10$

$\Rightarrow f(x) \leq 10$ [Using eq. (i)]

Therefore, maximum value of $f(x)$ is 10 which is obtained when $x-1=0$ i.e., $x=1$

And therefore, minimum value of $f(x)$ does not exist.

(iv) Given: $g(x) = x^3+1$

As $x \to \infty$ $g(x) \to \infty$

As $x \to -\infty \quad g(x) \to -\infty$

Therefore, maximum value and minimum value of $g(x)$ do not exist.

2. Find the maximum and minimum values, if any, of the following functions given by:

(i) $f(x) = |x+2| - 1$

(ii) $g(x) = -|x+1| + 3$

(iii) $h(x) = \sin(2x) + 5$

(iv) $f(x) = |\sin 4x + 3|$

(v) $h(x) = x + 1, x \in (-1, 1)$

Ans. (i) Given: $f(x) = |x+2| - 1$ ....(i)

Since $|x+2| \geq 0$ for all $x \in$ R

Subtracting 1 from both sides, $|x+2| - 2 \geq -1$

$\Rightarrow \quad f(x) \geq -1$

Therefore, minimum value of $f(x)$ is -1 which is obtained when $x + 2 = 0$ i.e., $x = -2$

From eq. (i), maximum value of $f(x) \to \infty$ hence it does not exist.

(ii) Given: $g(x) = -|x+1| + 3$

Since $|x+1| \geq 0$ for all $x \in$ R

Multiplying by -1 both sides and adding 3 both sides,

$|x+2| - 1 \geq -1$

$\Rightarrow \quad g(x) \le 3$

Therefore, maximum value of $g(x)$ is 3 which is obtained when $x+1=0$ i.e., $x=-1$

From eq. (i), minimum value of $g(x) \to -\infty$ hence it does not exist.

(iii) Given: $h(x)=\sin(2x)+5$ ....(i)

Since $-1 \le \sin 2x \le 1$ for all $x \in$ R

Adding 5 to all sides, $-1+5 \le \sin 2x+5 \le 1+5$

$\Rightarrow 4 \le h(x) \le 6$

Therefore, minimum value of $h(x)$ is 4 and maximum value is 6.

(iv) Given: $f(x)=|\sin 4x+3|$

Since $-1 \le \sin 4x \le 1$ for all $x \in$ R

Adding 3 to all sides, $-1+3 \le \sin 4x+3 \le 1+3$

$\Rightarrow 2 \le f(x) \le 4$

Therefore, minimum value of $f(x)$ is 2 and maximum value is 4.

(v) Given: $h(x)=x+1, x \in (-1,1)$ ....(i)

Since $-1 < x < 1$

Adding 1 to both sides, $-1+1 < x+1 < 1+1$

$\Rightarrow 0 < h(x) < 2$

Therefore, neither minimum value not maximum value of $h(x)$ exists.

3. Find the local maxima and local minima, if any, of the following functions. Find also

the local maximum and the local minimum values, as the case may be:

(i) $f(x)=x^2$

(ii) $g(x)=x^3-3x$

(iii) $h(x)=\sin x+\cos x, 0<x<\frac{\pi}{2}$

(iv) $f(x)=\sin x-\cos x, 0<x<2\pi$

(v) $f(x)=x^3-6x^2+9x+15$

(vi) $g(x)\frac{x}{2}+\frac{2}{x}, x>0$

(vii) $g(x)=\frac{1}{x^2+2}$

(viii) $f(x)=x\sqrt{1-x}, x>0$

Ans. (i) Given: $f(x)=x^2$

$\therefore\ f'(x)=2x$ and $f''(x)=2$

Now $f'(x)=0$

$\Rightarrow\ x=0$ [Turning point]

Again, when $x=0$, $f''(x)=2$ [Positive]

Therefore, $x=0$ is a point of local minima and local minimum value = $f(0)=(0)^2=0$

(ii) Given: $g(x)=x^3-3x$

$\therefore\ g'(x)=3x^2-3$ and $g''(x)=6x$

Now $g'(x)=0$

$\Rightarrow$ $3x^2-3=0$

$\Rightarrow 3(x^2-1)=0$

$\Rightarrow 3(x+1)(x-1)=0$

$\Rightarrow$ $x=-1$ or $x=1$ [Turning points]

Again, when $x=-1$,

$g''(x)=6x=6(-1)=-6$ [Negative]

$\therefore$ $x=-1$ is a point of local maxima and local maximum value
$g(-1)=(-1)^3-3(-1)=2$

And when $x=1$ $g''(x)=6x=6(1)=6$ [Positive]

$\therefore$ $x=1$ is a point of local minima and local minimum value $g(1)=(1)^3-3(1)=-2$

(iii) Given: $h(x)=\sin x+\cos x$ $\left(0<x<\frac{\pi}{2}\right)$ ....(i)

$\therefore$ $h'(x)=\cos x-\sin x$ and $h''(x)=-\sin x-\cos x$

Now $h'(x)=0$

$\Rightarrow \cos x-\sin x=0$

$\Rightarrow -\sin x=-\cos x$

$\Rightarrow \frac{\sin x}{\cos x}=1$

$\Rightarrow \tan x=1$ [Positive]

$\therefore$ x can have values in both I and III quadrant.

But, $\left(0 < x < \frac{\pi}{2}\right)$ therefore, x is only in I quadrant.

$\therefore$ $\tan x = 1$ = tan $\frac{\pi}{4}$

$\therefore$ x= $\frac{\pi}{4}$ [Turning point]

At $x = \frac{\pi}{4}$ $h''(x) = -\sin x - \cos x$

$\Rightarrow h''(x)$ = $-\sin\frac{\pi}{4} - \cos\frac{\pi}{4}$

= $\frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$ = $\frac{-2}{\sqrt{2}}$ = $-\sqrt{2}$ [Negative]

$\therefore$ $x = \frac{\pi}{4}$ is a point of local maxima and local maximum value

= $h\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$

= $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$

(iv) Given: $f(x) = \sin x - \cos x$ $(0 < x < 2\pi)$ ....(i)

$\therefore$ $f'(x) = \cos x + \sin x$ and $f''(x) = -\sin x + \cos x$

Now $f'(x) = 0$

$\Rightarrow \cos x + \sin x = 0$

$\Rightarrow \sin x = -\cos x$

$\Rightarrow \dfrac{\sin x}{\cos x} = -1$

$\Rightarrow \tan x = -1$ [Negative]

$\therefore$ x can have values in both II and IV quadrant.

$\therefore$ $\tan x = -1 = -\tan\dfrac{\pi}{4}$

$= \tan\left(\pi - \dfrac{\pi}{4}\right)$ or $\tan\left(2\pi - \dfrac{\pi}{4}\right)$

$\Rightarrow \tan x = \tan\dfrac{3\pi}{4}$ or $\tan\dfrac{7\pi}{4}$

$\Rightarrow x = \dfrac{3\pi}{4}$ and $x = \dfrac{7\pi}{4}$ [Turning point]

At $x = \dfrac{3\pi}{4}$ $f''(x) = -\sin x + \cos x = -\sin\dfrac{3\pi}{4} + \cos\dfrac{3\pi}{4}$

$\Rightarrow h''(x) = -\sin\left(\pi - \dfrac{\pi}{4}\right) + \cos\left(\pi - \dfrac{\pi}{4}\right)$

$= -\sin\dfrac{\pi}{4} - \cos\dfrac{\pi}{4}$

$= \dfrac{-1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}$

$= \dfrac{-2}{\sqrt{2}} = -\sqrt{2}$ [Negative]

$\therefore$ $x = \dfrac{3\pi}{4}$ is a point of local maxima and local maximum value =

$f\left(\dfrac{3\pi}{4}\right) = \sin\dfrac{3\pi}{4} - \cos\dfrac{3\pi}{4}$

$= \sin\left(\pi - \frac{\pi}{4}\right) - \cos\left(\pi - \frac{\pi}{4}\right)$

$= \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$ = $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$

At $x = \frac{7\pi}{4}$ $f''(x) = -\sin x + \cos x$

$= -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4}$

$\Rightarrow h''(x)$ = $-\sin\left(2\pi - \frac{\pi}{4}\right) + \cos\left(2\pi - \frac{\pi}{4}\right)$

$= \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

$= \frac{2}{\sqrt{2}}$ = $\sqrt{2}$ [Positive]

$\therefore$ $x = \frac{7\pi}{4}$ is a point of local maxima and local maximum value =

$f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4}$

$= \sin\left(2\pi - \frac{\pi}{4}\right) - \cos\left(2\pi - \frac{\pi}{4}\right)$

$= -\sin\frac{\pi}{4} - \cos\frac{\pi}{4}$ = $\frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$

(v) Given: $f(x) = x^3 - 6x^2 + 9x + 15$

$\therefore \quad f'(x) = 3x^2 - 12x + 9$ and $f''(x) = 6x - 12$

Now $f'(x) = 0$

$\Rightarrow 3x^2 - 12x + 9 = 0$

$\Rightarrow x^2 - 4x + 3 = 0$

$\Rightarrow (x-1)(x-3) = 0$

$\Rightarrow x = 1$ or $x = 3$ [Turning points]

At $x = 1$, $f''(x) = 6x - 12 = 6 - 12 = -6$ [Negative]

$\therefore \quad x = 1$ is a point of local maxima and local maximum value is
$f(1) = (1)^3 - 6(1)^2 + 9(1) + 15 = 19$

At $x = 3$, $f''(x) = 6x - 12 = 6 \times 3 - 12 = 6$ [Positive]

$\therefore \quad x = 3$ is a point of local minima and local minimum value is
$f(3) = (3)^3 - 6(3)^2 + 9(3) + 15 = 15$

(vi) Given: $g(x) = \frac{x}{2} + \frac{2}{x}, x > 0$

$\therefore \quad g'(x) = \frac{1}{2} - \frac{2}{x^2}$

$= \frac{x^2 - 4}{2x^2}$

$= \frac{(x+2)(x-2)}{2x^2}$ and $g''(x) = \frac{4}{x^3}$

Now $g'(x) = 0$

$\Rightarrow \dfrac{(x+2)(x-2)}{2x^2} = 0$

$\Rightarrow (x+2)(x-2)=0$

$\Rightarrow x=-2$ or $x=2$

But $x>0$, therefore $x=2$ is only the turning point.

$\therefore$ $x=2$ is a point of local minima and local minimum value is $g(2)=\dfrac{2}{2}+\dfrac{2}{2}=2$

(vii) Given: $h(x)=\dfrac{1}{x^2+2}=\left(x^2+2\right)^{-1}$

$\therefore$ $h'(x)=(-1)\left(x^2+2\right)^{-2}(2x)=\dfrac{-2x}{\left(x^2+2\right)^2}$ and

$h''(x)=\frac{(x^2+2)^2(-2)-(-2x)(2)(x^2+2)2x}{(x^2+2)^4}$

$=\frac{(x^2+2)\left[-2(x^2+2)+8x^2\right]}{(x^2+2)^4}$

$=\frac{-2x^2-4+8x^2}{(x^2+2)^3}$

$= \dfrac{-2\left(2-3x^2\right)}{\left(x^2+2\right)^3}$

Now $h'(x)=0$

$\Rightarrow \dfrac{-2x}{\left(x^2+2\right)^2}=0$

$\Rightarrow x=0$ [Turning point]

At $x=0$, $h''(x)=\frac{-2(2-3x^2)}{(x^2+2)^3}=\frac{-2(2-0)}{(0+2)^3}=\frac{-4}{8}=\frac{-1}{2}$ [Negative]

$\therefore$ $x=0$ is a point of local maxima and local maximum value is $h(0)=\frac{1}{0+2}=\frac{1}{2}$

(viii) Given: $f(x)=x\sqrt{1-x}, x>0$

$\therefore$ $f'(x)=x.\frac{1}{2}(1-x)^{\frac{-1}{2}}\frac{d}{dx}(1-x)+\sqrt{1-x}.1$

= $\frac{-x}{2\sqrt{1-x}}+\sqrt{1-x}$

= $\frac{-x+2(1-x)}{2\sqrt{1-x}}$ = $\frac{2-3x}{2\sqrt{1-x}}$

And $f''(x)=\frac{1}{2}.\frac{\sqrt{1-x}.(-3)-(2-3x).\frac{1}{2\sqrt{1-x}}(-1)}{1-x}$

= $\frac{-6(1-x)+2-3x}{4(1-x)^{\frac{3}{2}}}$

= $\frac{3x-4}{4(1-x)^{\frac{3}{2}}}$

Now $f'(x)=0$

$\Rightarrow$ $\frac{2-3x}{2\sqrt{1-x}}$ = 0

$\Rightarrow$ $2-3x=0$

$\Rightarrow\ x = \frac{2}{3}$ turning point

Again $f''\left(\frac{2}{3}\right) = \dfrac{3\left(\frac{2}{3}\right) - 4}{4\left(1 - \frac{2}{3}\right)^{\frac{3}{2}}}$

$= \dfrac{2-4}{4\left(\frac{1}{3}\right)^{\frac{3}{2}}} = \dfrac{-1}{2\left(\frac{1}{3}\right)^{\frac{3}{2}}} < 0$

$\therefore\ x = \frac{2}{3}$ is a point of local maxima and local maximum value is

$f\left(\frac{2}{3}\right) = x\sqrt{1-x} = \frac{2}{3}\sqrt{1 - \frac{2}{3}} = \frac{2\sqrt{3}}{9}$

$\therefore\ f(x)$ has local maximum value at $x = \frac{2}{3}$.

6. Find the maximum profit that a company can make, if the profit function is given by $p(x) = 41 + 24x - 18x^2$.

Ans. Given: Profit function $p(x) = 41 + 24x - 18x^2$

$\therefore\ p'(x) = 24 - 36x$ and $p''(x) = -36$

Now $p'(x) = 0$

$\Rightarrow 24 - 36x = 0$

$\Rightarrow x = \frac{24}{36} = \frac{2}{3}$

At $x=\frac{2}{3}$, $p''(x)=-36$ [Negative]

$\therefore$ $p(x)$ has a local maximum value at $x=\frac{2}{3}$.

$\therefore$ At $x=\frac{2}{3}$, Maximum profit

= $41+24\left(\frac{2}{3}\right)-18\left(\frac{4}{9}\right)$

= 41 + 16 – 8 = 49

7. Find both the maximum value and minimum value of $3x^4-8x^3+12x^2-48x+25$ on the interval [0, 3].

Ans. Let $f(x)=3x^4-8x^3+12x^2-48x+25$ on [0, 3]

$\therefore$ $f'(x)=12x^3-24x^2+24x-48$

Now $f'(x)=0$

$\Rightarrow 12x^3-24x^2+24x-48=0$

$\Rightarrow x^3-2x^2+2x-4=0$

$\Rightarrow (x-2)(x^2+2)=0$

$\Rightarrow x=2$ or $x=\pm\sqrt{2}$

Since $x=\pm\sqrt{2}$ is imaginary, therefore it is rejected.

$\therefore$ $x=2$ is turning point.

$\therefore$ At $x=2$, $f(2)=3(16)-8(8)+12(4)-48(2)+25$ = -39

At $x=0$ $f(0)=25$

At $x=3$, $f(3)=3(81)-8(27)+12(9)-48(3)+25=16$

Therefore, absolute minimum value is -39 and absolute maximum value is 25.

8. At what points on the interval $[0, 2\pi]$ does the function $\sin 2x$ attain its maximum value?

Ans. Let $f(x)=\sin 2x$

$\Rightarrow$ $f'(x)=2\cos 2x$

Now $f'(x)=0$

$\Rightarrow$ $2\cos 2x=0$

$\Rightarrow$ $2x=(2n+1)\frac{\pi}{2}$

$\Rightarrow$ $x=(2n+1)\frac{\pi}{4}$

Putting $n=0, 1, 2, 3$; $x=\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \in [0, 2\pi]$

Now $f(x)=\sin 2x$

$\therefore$ $f\left[(2n+1)\frac{\pi}{4}\right]=\sin (2n+1)\frac{\pi}{2}$

$= \sin\left(n\pi+\frac{\pi}{2}\right)$

$= (-1)^n \sin\frac{\pi}{2} = (-1)^n$

Putting $n = 0, 1, 2, 3;$

$$f\left(\frac{\pi}{4}\right) = (-1)^0 = 1$$

$$f\left(\frac{3\pi}{4}\right) = (-1)^1 = -1$$

$$f\left(\frac{5\pi}{4}\right) = (-1)^2 = 1$$

$$f\left(\frac{7\pi}{4}\right) = (-1)^3 = -1$$

Also $f(0) = \sin 0 = 0$ and $f(2\pi) = \sin 4\pi = 0$

Since $f(x)$ attains its maximum value 1 at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

Therefore, the required points are $\left(\frac{\pi}{4}, 1\right)$ and $\left(\frac{5\pi}{4}, 1\right)$.

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9. What is the maximum value of the function $\sin x + \cos x$?

Ans. Let $f(x) = \sin x + \cos x$

$\Rightarrow f'(x) = \cos x - \sin x$

Now $f'(x) = 0$

$\Rightarrow \cos x - \sin x = 0$

$\Rightarrow -\sin x = -\cos x$

$\tan x = 1 = \tan \frac{\pi}{4}$

$\Rightarrow x = n\pi + \frac{\pi}{4}$ [Turning point]

$\therefore f\left(n\pi + \frac{\pi}{4}\right) = \sin\left(n\pi + \frac{\pi}{4}\right) + \cos\left(n\pi + \frac{\pi}{4}\right)$

= $(-1)^n \sin\frac{\pi}{4} + (-1)^n \cos\frac{\pi}{4}$

= $(-1)^n \frac{1}{\sqrt{2}} + (-1)^n \frac{1}{\sqrt{2}}$

= $2(-1)^n . \frac{1}{\sqrt{2}}$

= $\sqrt{2}(-1)^n$

If n is even, then $f\left(n\pi + \frac{\pi}{4}\right) = \sqrt{2}$

If n is odd, then $f\left(n\pi + \frac{\pi}{4}\right) = -\sqrt{2}$

Therefore, maximum value of $f(x)$ is $\sqrt{2}$ and minimum value of $f(x)$ is $-\sqrt{2}$.

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10. Find the maximum value of $2x^3 - 24x + 107$ in the interval [1, 3]. Find the maximum value of the same function in $[-3, -1]$.

Ans. Let $f(x) = 2x^3 - 24x + 107$

$\Rightarrow f'(x) = 6x^2 - 24$

Now $f'(x) = 0$

$\Rightarrow 6x^2 - 24 = 0$

$\Rightarrow\ x^2 = 4$

$\Rightarrow\ x = \pm 2$

$\Rightarrow\ x = 2$ or $x = -2$ [Turning points]

For Interval [1, 3], $x = 2$ is turning point.

At $x = 1$, $f(1) = 2(1) - 24(1) + 107 = 85$

At $x = 2$, $f(2) = 2(8) - 24(2) + 107 = 75$

At $x = 3$, $f(3) = 2(27) - 24(3) + 107 = 89$

Therefore, maximum value of $f(x)$ is 89.

For Interval $[-3, -1]$, $x = -2$ is turning point.

At $x = -1$, $f(1) = 2(-1) - 24(-1) + 107 = 129$

At $x = -2$, $f(2) = 2(-8) - 24(-2) + 107 = 139$

At $x = -3$, $f(3) = 2(-27) - 24(-3) + 107 = 125$

Therefore, maximum value of $f(x)$ is 139.

11. It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval [0, 20]. Find the value of a.

Ans. Let $f(x) = x^4 - 62x^2 + ax + 9$

$\Rightarrow\ f'(x) = 4x^3 - 124x + a$

Since, $f(x)$ attains its maximum value at $x = 1$ in the interval [0, 2], therefore $f'(1) = 0$

$\therefore\ f'(1) = 4 - 124 + a = 0$

$\Rightarrow\ a-120=0$

$\Rightarrow\ a=120$

12. Find the maximum and minimum value of $x+\sin x$ **on** $[0,2\pi]$.

Ans. Let $f(x)=x+\sin 2x$

$\Rightarrow\ f'(x)=1+2\cos 2x$

Now $f'(x)=0$

$\Rightarrow\ 1+2\cos 2x=0$

$\Rightarrow\ 2\cos 2x=-1$

$\Rightarrow\ \cos 2x=\dfrac{-1}{2}$

$= -\cos\dfrac{\pi}{3} = \cos\left(\pi-\dfrac{\pi}{3}\right)$

$= \cos\dfrac{2\pi}{3}$

$\Rightarrow\ 2x=2n\pi\pm\dfrac{2\pi}{3}$ where $n\in$ Z $\Rightarrow\ x=n\pi\pm\dfrac{\pi}{3}$

For $n=0$, $x=\pm\dfrac{\pi}{3}$ But $x=-\dfrac{\pi}{3}\notin[0,2\pi]$, therefore $x=\dfrac{\pi}{3}$

For $n=1$, $x=\pi\pm\dfrac{\pi}{3}$ = $\pi+\dfrac{\pi}{3}$ and $\pi-\dfrac{\pi}{3}$

For $n=2$, $x=2\pi\pm\dfrac{\pi}{3}$

But $x = 2\pi + \frac{\pi}{3} > 2\pi \notin [0, 2\pi]$, therefore $x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

Therefore, it is clear that the only turning point of $f(x)$ given by $x + \sin 2x$ which belong to given closed interval $[0, 2\pi]$ are, $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

At $x = \frac{\pi}{3}$

$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2} = 1.05 + 0.87 = 1.92$ nearly

At $x = \frac{2\pi}{3}$

$f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sin\frac{4\pi}{3} = 2\pi - \frac{\sqrt{3}}{2} = 2.10 - 0.87 = 1.23$ nearly

At $x = \frac{4\pi}{3}$

$f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} + \sin\frac{8\pi}{3} = \frac{4\pi}{3} + \frac{\sqrt{3}}{2} = 4 \times 1.05 + 0.87 = 5.07$ nearly

At $x = \frac{5\pi}{3}$

$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sin\frac{10\pi}{3} = \frac{5\pi}{3} - \frac{\sqrt{3}}{2} = 5 \times 1.05 - 0.87 = 4.38$ nearly

At $x = 0$ $f(0) = 0 + \sin 0 = 0$

At $x = 2\pi$

$f(2\pi) = 2\pi + \sin 4\pi = 2\pi + 0 = 2\pi = 2 \times 3.14 = 6.28$ nearly

Therefore, Maximum value = 2π and minimum value = 0

13. Find two numbers whose sum is 24 and whose product is as large as possible.

Ans. Let the two numbers be x and y.

According to the question, $x+y=24$

$\Rightarrow y=24-x$ ....(i)

And let z is the product of x and y.

$\Rightarrow z=xy$

$\Rightarrow z=x(24-x)$ [From eq. (i)]

$\Rightarrow z=24x-x^2$

$\Rightarrow \frac{dz}{dx}=24-2x$ and $\frac{d^2z}{dx^2}=-2$

Now to find turning point, $\frac{dz}{dx}=0$

$\Rightarrow 24-2x=0 \Rightarrow x=12$

At $x=12$, $\frac{d^2z}{dx^2}=-2$ [Negative]

$\therefore$ $x=12$ is a point of local maxima and z is maximum at $x=12$.

$\therefore$ From eq. (i), $y=24-12=12$

Therefore, the two required numbers are 12 and 12.

14. Find two positive integers x and y such that $x+y=60$ and xy^3 is maximum.

Ans. Given: $x+y=60, x>0, y>0$ ....(i)

Let P = xy^3 [To be maximized] ....(ii)

Putting from eq. (i), $x = 60 - y$ in eq. (ii),

P = $(60 - y)y^3 = 60y^3 - y^4$

$\Rightarrow \frac{dP}{dy} = 180y^2 - 4y^3 = 4y^2(45 - y)$ .....(iii)

Now $\frac{dP}{dy} = 0$

$\Rightarrow 4y^2(45 - y) = 0$

$\Rightarrow y = 0, 45$

It is clear that $\frac{dP}{dy}$ changes sign from positive to negative as y increases through 45.

Therefore, P is maximum when $y = 45$.

Hence, xy^3 is maximum when $x = 60 - 45 = 15$ and $y = 45$.

---

15. Find two positive integers x and y such that their sum is 35 and the product x^2y^5 is a maximum.

Ans. Given: $x + y = 35$

$\Rightarrow y = 35 - x$ ....(i)

Let $z = x^2y^5$

$\Rightarrow x^2(35 - x)^5$ [From eq. (i)]

$\Rightarrow \frac{dz}{dx} = x^2.5(35 - x)^4(-1) + (35 - x)^5 2x$

$\Rightarrow \frac{dz}{dx} = x(35-x)^4[-5x+(35-x)2]$

$\Rightarrow \frac{dz}{dx} = x(35-x)^4[-5x+70-2x]$

$\Rightarrow \frac{dz}{dx} = x(35-x)^4(70-7x)$

$\Rightarrow \frac{dz}{dx} = 7x(35-x)^4(10-x)$ ....(ii)

Now $\frac{dz}{dx} = 0$

$\Rightarrow 7x(35-x)^4(10-x) = 0$

$\Rightarrow x = 0$ or $35 - x = 0$ or $10 - x = 0$

$\Rightarrow x = 0$ or $x = 35$ or $x = 10$

Now $x = 0$ is rejected because according to question, x is a positive number.

Also $x = 35$ is rejected because from eq. (i), y = 35 – 35 = 0, but y is positive.

Therefore, $x = 10$ is only the turning point.

$\therefore \frac{d^2z}{dx^2} = 7(35-x)^3(6x^2 - 120x + 350)$

At $x = 10$, $\frac{d^2z}{dx^2} = 7(35-10)^3(6\times100 - 120\times10 + 350)$

= $7(25)^3(-250) < 0$

$\therefore$ By second derivative test, $\frac{dz}{dx}$ will be maximum at $x = 10$ when $y = 35 - 10 = 25$.

Therefore, the required numbers are 10 and 25.

16. Find two positive integers whose sum is 16 and sum of whose cubes is minimum.

Ans. Let the two positive numbers are x and y.

$\therefore \quad x+y=16$

$\Rightarrow y=16-x$ ....(i)

Let $z=x^3+y^3$

$\Rightarrow z=x^3+(16-x)^3$ [From eq. (i)]

$\Rightarrow z=x^3+(16)^3-x^3-48x(16-x)$

= $(16)^3-768x+48x^2$

$\Rightarrow \frac{dz}{dx}=-768+96x$ and $\frac{d^2z}{dx^2}=96$

Now $\frac{dz}{dx}$ = 0

$\Rightarrow -768+96x=0$

$\Rightarrow x=8$

At $x=8$ $\frac{d^2z}{dx^2}=96$ is positive.

$\therefore \quad x=8$ is a point of local minima and z is minimum when $x=8$.

$\therefore \quad y=16-8=8$

Therefore, the required numbers are 8 and 8.

17. A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the

side of the square to be cut off so that the volume of the box is the maximum possible?

Ans. Given: Each side of square piece of tin is 18 cm.

Let x cm be the side of each of the four squares cut off from each corner.

Then dimensions of the open box formed by folding the flaps after cutting off squares are $(18-2x), (18-2x)$ and x cm.

Let z denotes the volume of the open box.

$\therefore \quad z=(18-2x)(18-2x)x$

$\Rightarrow \quad z=(18-2x)^2 x$

$\Rightarrow \quad z=(324+4x^2-72x)x$

$= 4x^3-72x^2+324x$

$\Rightarrow \quad \frac{dz}{dx}=12x^2-144x+324$ and $\frac{d^2z}{dx^2}=24x-144$

Now $\frac{dz}{dx}$ = 0

$\Rightarrow 12x^2-144x+324=0$

$= x^2-12x+27=0$

$\Rightarrow (x-9)(x-3)=0$

$\Rightarrow x=9$ or $x=3$

$x=9$ is rejected because at $x=9$ length = $18-2x=18-2\times 9=0$ which is impossible.

$\therefore \quad x=3$ is the turning point.

At $x=3$, $\frac{d^2z}{dx^2}=24\times 3-144=-72$ [Negative]

$\therefore$ z is minimum at $x=3$ i.e., side of each square to be cut off from each corner for maximum volume is 3 cm.

18. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum?

Ans. Given: Dimensions of rectangular sheet are 45 cm and 24 cm.

Let x cm be the side of each of the four squares cut off from each corner.

Then dimensions of the open box formed by folding the flaps after cutting off squares are $(45-2x)$, $(24-2x)$ and x cm.

Let z denotes the volume of the open box.

$\therefore \quad z=(45-2x)(24-2x)\,x$

$\Rightarrow \quad z=(1080-138x+4x^2)\,x$

$= 4x^3-138x^2+1080x$

$\Rightarrow \quad \frac{dz}{dx}=12x^2-276x+1080$ and $\frac{d^2z}{dx^2}=24x-276$

Now $\frac{dz}{dx}$ = 0

$\Rightarrow 12x^2-276x+1080=0$

$\Rightarrow =x^2-23x+90=0$

$\Rightarrow (x-5)(x-18)=0$

$\Rightarrow x=5$ or $x=18$

$x=18$ is rejected because at $x=18$ length = $24-2x=18-2\times18=-12$ which is impossible.

$\therefore \quad x=5$ is the turning point.

At $x=5$, $\dfrac{d^2z}{dx^2}=24\times3-276=-156$ [Negative]

· z is minimum at $x=5$ i.e., side of each square to be cut off from each corner for maximum volume is 5 cm.

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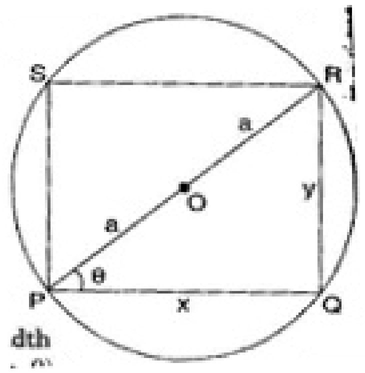
19. Show that of all the rectangles inscribed in a given fixed circle, the square has maximum area.

Ans. Let PQRS be the rectangle inscribed in a given circle with centre O and radius a.



Let x and y be the length and breadth of the rectangle, i.e., $x>0$ and $y>0$.

In right angled triangle PQR, using Pythagoras theorem,

$PQ^2 + QR^2 = PR^2$

$\Rightarrow \quad x^2+y^2=(2a)^2$

$\Rightarrow \quad y^2=4a^2-x^2$

$\Rightarrow \quad y=\sqrt{4a^2-x^2}$ .....(i)

Let A be the area of the rectangle, then A = xy = $x\sqrt{4a^2-x^2}$

$$\Rightarrow \quad \frac{dA}{dx}=\sqrt{4a^2-x^2}+x\frac{1}{2\sqrt{4a^2-x^2}}(-2x)=\sqrt{4a^2-x^2}-\frac{x^2}{\sqrt{4a^2-x^2}}$$

= $\dfrac{4a^2-2x^2}{\sqrt{4a^2-x^2}}$

And $\dfrac{d^2\text{A}}{dx^2}=\dfrac{\sqrt{4a^2-x^2}\,(-4x)-\left(4a^2-2x^2\right)\dfrac{(-2x)}{2\sqrt{4a^2-x^2}}}{\left(4a^2-2x^2\right)}$

= $\dfrac{\left(4a^2-2x^2\right)(-4x)+x\left(4a^2-2x^2\right)}{\left(4a^2-2x^2\right)^{\frac{3}{2}}}$

$\Rightarrow \dfrac{d^2\text{A}}{dx^2}=\dfrac{-12a^2x+2x^3}{\left(4a^2-2x^2\right)^{\frac{3}{2}}}$

= $\dfrac{-2x\left(6a^2-x^2\right)}{\left(4a^2-2x^2\right)^{\frac{3}{2}}}$

Now $\dfrac{d\text{A}}{dx}=0$

$\Rightarrow \dfrac{4a^2-2x^2}{\sqrt{4a^2-x^2}}$ = 0

$\Rightarrow 4a^2-2x^2=0$

$\Rightarrow x=\sqrt{2}a$

$\therefore$ At $x=\sqrt{2}a$, $\dfrac{d^2\text{A}}{dx^2}=\dfrac{-2\left(\sqrt{2}a\right)\left(6a^2-2a^2\right)}{2\sqrt{2}a^3}=\dfrac{-8\sqrt{2}a^3}{2\sqrt{2}a^3}=-4$ [Negative]

$\therefore$ At $x=\sqrt{2}a$, area of rectangle is maximum.

And from eq. (i), $y=\sqrt{4a^2-2a^2}=\sqrt{2}a$, i.e., $x = y = \sqrt{2}a$

Therefore, the area of inscribed rectangle is maximum when it is square.

20. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

Ans. Let x be the radius of the circular base and y be the height of closed right circular cylinder.

$\therefore$ Total surface area (S) = $2\pi xy + 2\pi x^2$

$\Rightarrow xy + x^2 = \frac{S}{2\pi}$ = k (say)

$\Rightarrow xy = k - x^2$

$\Rightarrow y = \frac{k - x^2}{x}$ .....(i)

Volume of cylinder $(z) = \pi x^2 y$

= $\pi x^2 \left(\frac{k - x^2}{x}\right)$ [From eq. (i)]

$\Rightarrow z = \pi x(k - x^2) = \pi(kx - x^3)$

$\Rightarrow \frac{dz}{dx} = \pi(k - 3x^2)$ and $\frac{d^2z}{dx^2} = \pi(-6x) = -6\pi x$

Now $\frac{dz}{dx} = 0$

$\Rightarrow \pi(k - 3x^2) = 0$

$\Rightarrow\ x=\sqrt{\dfrac{k}{3}}$

At $x=\sqrt{\dfrac{k}{3}}$ $\dfrac{d^2z}{dx^2}=-6\pi\sqrt{\dfrac{k}{3}}$ [Negative]

$\therefore$ z is maximum at $x=\sqrt{\dfrac{k}{3}}$.

$\therefore$ From eq. (i), $y=\dfrac{k-\dfrac{k}{3}}{\sqrt{\dfrac{k}{3}}}$

= $2\sqrt{\dfrac{k}{3}}$ = $2x$

$\Rightarrow$ Height = Diameter

Therefore, the volume of cylinder is maximum when its height is equal to the diameter of its base.

---

21. Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimeters, find the dimensions of the can which has the minimum surface area.

Ans. Let x be the radius of the circular base and y be the height of closed right circular cylinder.

According to the question, Volume of the cylinder $\pi x^2 y=100$

$\Rightarrow\ y=\dfrac{100}{\pi x^2}$ ...(i)

$\therefore$ Total surface area (S) = $2\pi xy+2\pi x^2$

= $2\pi\left(xy+x^2\right)$

= $2\pi\left(x.\frac{100}{\pi x^2}+x^2\right)$ [From eq. (i)]

$\Rightarrow$ S = $2\pi\left(\frac{100}{\pi x}+x^2\right)$

= $2\pi\left(\frac{100}{\pi}x^{-1}+x^2\right)$

$\Rightarrow \frac{dS}{dx}=2\pi\left(-\frac{100}{\pi}x^{-2}+2x\right)$ and $\frac{d^2S}{dx^2}=2\pi\left(\frac{200}{\pi}x^{-3}+2\right)$

Now $\frac{dS}{dx}=0$

$\Rightarrow 2\pi\left(-\frac{100}{\pi}x^{-2}+2x\right)=0$

$\Rightarrow \left(-\frac{100}{\pi}x^{-2}+2x\right)=0$

$\Rightarrow \frac{100}{\pi}x^{-2}=2x$

$\Rightarrow x^3=\frac{100}{2\pi}=\frac{50}{\pi}$

$\Rightarrow x=\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$

At $x=\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ $\frac{d^2S}{dx^2}=2\pi\left(\frac{200}{\pi\left(\frac{50}{\pi}\right)}+2\right)$

= $2\pi(4+2)$ = 12π [Positive]

$\therefore$ S is minimum when radius $x = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm

$\therefore$ From eq. (i)

$$y = \frac{100}{x\left(\frac{50}{\pi}\right)^{\frac{2}{3}}}$$

= $2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ = $2x$

22. A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

Ans. Let x meters be the side of square and y meters be the radius of the circle.

Length of the wire = Perimeter of square + Circumference of circle

$\Rightarrow$ $4x+2\pi y$ = 28

$\Rightarrow$ $2x+\pi y=14$

$\Rightarrow$ $y = \frac{14-2x}{\pi}$ ....(i)

Area of square = x^2 and Area of circle = πy^2

Combined area (A) = x^2 + πy^2 = x^2 + $\pi\left(\frac{14-2x}{\pi}\right)^2$

= $x^2 + \frac{4}{\pi}(7-x)^2$

$\Rightarrow \dfrac{dA}{dx} = 2x - \dfrac{8}{\pi}(7-x)$ and $\dfrac{d^2A}{dx^2} = 2 + \dfrac{8}{\pi}$

Now $\dfrac{dA}{dx} = 0$

$\Rightarrow 2x - \dfrac{8}{\pi}(7-x) = 0$

$\Rightarrow 2x = \dfrac{8}{\pi}(7-x)$

$\Rightarrow 2\pi x = 56 - 8x$

$\Rightarrow (2\pi + 8)x = 56$

$\Rightarrow x = \dfrac{56}{2\pi+8} = \dfrac{28}{\pi+4}$

And $\dfrac{d^2A}{dx^2} = 2 + \dfrac{8}{\pi}$ [Positive]

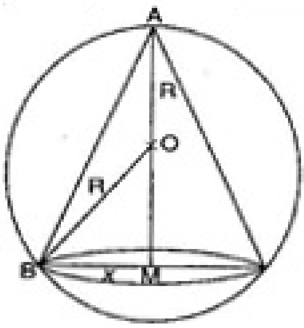
$\therefore$ A is minimum when $x = \dfrac{28}{\pi+4}$

Therefore, the wire should be cut at a distance $4x = \dfrac{28}{\pi+4}$ from one end.

---

23. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\dfrac{8}{27}$ of the volume of the sphere.

Ans. Let O be the centre and R be the radius of the given sphere, BM = x and AM = y



In right angled triangle OMB, using Pythagoras theorem,

$OM^2 + BM^2 = OB^2$

$\Rightarrow (y-R)^2 + x^2 = R^2$

$\Rightarrow y^2 + R^2 - 2Ry + x^2 = R^2$

$\Rightarrow y^2 - 2Ry + x^2 = 0$

$\Rightarrow x^2 = 2Ry - y^2$ ....(i)

Volume of a cone inscribed in the given sphere

$(z) = \frac{1}{3}\pi x^2 y = \frac{1}{3}\pi\left(2Ry - y^2\right)y$

$\Rightarrow z = \frac{\pi}{3}\left(2Ry^2 - y^3\right)$ ....(ii)

$\Rightarrow \frac{dz}{dx} = \frac{\pi}{3}\left(4Ry - 3y^2\right)$ and $\frac{d^2z}{dx^2} = \frac{\pi}{3}\left(4R - 6y\right)$

Now $\frac{dz}{dx} = 0$

$\Rightarrow \frac{\pi}{3}\left(4Ry - 3y^2\right) = 0$

$\Rightarrow 4Ry - 3y^2 = 0$

$\Rightarrow 3y^2 = 4Ry$

$\Rightarrow y = \frac{4R}{3}$

At $y = \frac{4R}{3}$ $\frac{d^2z}{dx^2} = \frac{\pi}{3}\left(4R - 6.\frac{4R}{3}\right)$

= $\frac{\pi}{3}(4R - 8R)$

= $\frac{-4R}{3}$ [Negative]

$\therefore$ z is maximum at $y = \frac{4R}{3}$

$\therefore$ From eq. (i) $x^2 = 2R.\frac{4R}{3} - \left(\frac{4R}{3}\right)^2 = \frac{8R^2}{3} - \frac{16R^2}{9}$

= $\frac{8R^2}{9}$

$\therefore$ Maximum volume of the cone

= $\frac{1}{3}\pi x^2 y = \frac{1}{3}\pi.\frac{8R^2}{9}.\frac{4R}{3} = \frac{8}{27}.\frac{4}{3}\pi R^3$

= $\frac{8}{27}$ (Volume of the sphere)

24. Show that the right circular cone of least curve surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.

Ans. Let x be the radius and y be the height of the cone.

$\therefore$ Volume of the cone (V) = $\frac{1}{3}\pi x^2 y$

$\Rightarrow x^2 y = \frac{3V}{\pi} = k$ (say) ....(i)

And Surface area of the cone (S) = $\pi x\sqrt{x^2 + y^2}$

$\Rightarrow S^2 = \pi^2 x^2 \left(x^2 + y^2\right) = z$ (say) ....(ii)

$\Rightarrow z = \pi^2 . \frac{k}{y}\left(\frac{k}{y} + y^2\right)$

= $\pi^2 k\left(\frac{k}{y^2} + y\right)$

= $\pi^2 k\left(ky^{-2} + y\right)$

$\Rightarrow \frac{dz}{dy} = \pi^2 k\left[-2ky^{-3} + 1\right]$ and $\frac{d^2 z}{dy^2} = \pi^2 k\left[6ky^{-4}\right] = \frac{6\pi^2 k^2}{y^4}$

Now $\frac{dz}{dy} = 0$

$\Rightarrow \pi^2 k\left[-2ky^{-3} + 1\right] = 0$

$\Rightarrow \frac{-2k}{y^3} + 1 = 0$

$\Rightarrow \frac{2k}{y^3} = 1$

$\Rightarrow y^3 = 2k$

$\Rightarrow y = (2k)^{\frac{1}{3}}$ ....(iii)

At $y=(2k)^{\frac{1}{3}}$ $\frac{d^2z}{dy^2}=\frac{6\pi^2k^2}{(2k)^{\frac{4}{3}}}$ [Positive]

$\therefore$ z is minimum when $y=(2k)^{\frac{1}{3}}$

$\therefore$ From eq. (i), $x^2=\frac{k}{y}=\frac{k}{(2k)^{\frac{1}{3}}}$

= $\frac{2k}{2(2k)^{\frac{1}{3}}}=\frac{(2k)^{\frac{2}{3}}}{2}=\frac{y^2}{2}$ [From eq. (iii)]

$\Rightarrow$ $y^2=2x^2$

$\Rightarrow$ $y=\sqrt{2}x$

Therefore, Surface area is minimum when height = $\sqrt{2}$ (radius of base)

---

25. Show that the semi-vertical angle of the cone of the maximum value and of given slant height is $\tan^{-1}\sqrt{2}$.

Ans. Let x be the radius, y be the height, l be the slant height of given cone and θ be the semi-vertical angle of cone.

$\therefore$ $l^2=x^2+y^2$

$\Rightarrow$ $x^2=l^2-y^2$ ....(i)

$\therefore$ Volume of the cone (V) = $\frac{1}{3}\pi x^2y$ ....(ii)

$\Rightarrow$ V = $\frac{1}{3}\pi\left(l^2-y^2\right)y$

= $\frac{\pi}{3}\left(l^2 y - y^3\right)$

$\Rightarrow \frac{dV}{dy} = \frac{\pi}{3}\left(l^2 - 3y^2\right)$ and $\frac{d^2V}{dy^2} = \frac{\pi}{3}(-6y) = -2\pi y$

Now $\frac{dV}{dy} = 0$

$\Rightarrow \frac{\pi}{3}\left(l^2 - 3y^2\right) = 0$

$\Rightarrow l^2 - 3y^2 = 0$

$\Rightarrow 3y^2 = l^2$

$\Rightarrow y = \frac{l}{\sqrt{3}}$

At $y = \frac{l}{\sqrt{3}}$ $\frac{d^2V}{dy^2} = -2\pi\left(\frac{l}{\sqrt{3}}\right)$

= $\frac{-2\pi l}{\sqrt{3}}$ [Negative]

$\therefore$ V is maximum at $y = \frac{l}{\sqrt{3}}$

$\therefore$ From eq. (i), $x^2 = l^2 - \frac{l^2}{3} = \frac{2l^2}{3}$

$\Rightarrow x = \sqrt{2}\frac{l}{\sqrt{3}}$

$\therefore$ Semi-vertical angle, $\tan\theta = \frac{x}{y}$

= $\dfrac{\sqrt{2}\dfrac{l}{\sqrt{3}}}{\dfrac{l}{\sqrt{3}}} = \sqrt{2}$

$\Rightarrow \quad \theta = \tan^{-1}\sqrt{2}$

26. Show that the semi-vertical angle of the right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\dfrac{1}{3}\right)$.

Ans. Let x be the radius and y be the height of the cone and semi-vertical angle be θ.

$\therefore$ Total Surface area of cone (S) = $\pi x\sqrt{x^2+y^2}+\pi x^2$

$\Rightarrow \quad x\sqrt{x^2+y^2}+x^2 = \dfrac{S}{\pi} = k$ (say)

$\Rightarrow \quad x\sqrt{x^2+y^2} = k - x^2$

$\Rightarrow \quad x^2\left(x^2+y^2\right) = \left(k-x^2\right)^2$

$\Rightarrow \quad x^4 + x^2y^2 = k^2 + x^4 - 2kx^2$

$\Rightarrow \quad x^2y^2 = k^2 - 2kx^2$

$\Rightarrow \quad x^2 = \dfrac{k^2}{y^2+2k}$ ....(i)

Volume of cone (V) = $\dfrac{1}{3}\pi x^2 y$

= $\dfrac{1}{3}\pi\left(\dfrac{k^2}{y^2+2k}\right)y$

= $\frac{1}{3}\pi k^2\left(\frac{y}{y^2+2k}\right)$

$\Rightarrow \frac{dV}{dy}=\frac{1}{3}\pi k^2\frac{d}{dy}.\frac{y}{y^2+2k}$

= $\frac{1}{3}\pi k^2\left[\frac{\left(y^2+2k\right).1-y.2y}{\left(y^2+2k\right)^2}\right]$ [Using quotient rule]

$\Rightarrow \frac{dV}{dy}=\frac{1}{3}\pi k^2\frac{\left(2k-y^2\right)}{\left(y^2+2k\right)^2}$ ....(ii)

Now $\frac{dV}{dy}=0$

$\Rightarrow \frac{1}{3}\pi k^2\frac{\left(2k-y^2\right)}{\left(y^2+2k\right)^2}=0$

$\Rightarrow 2k-y^2=0$

$\Rightarrow y^2=2k$

$\Rightarrow y=\pm\sqrt{2k}$

$\Rightarrow y=\sqrt{2k}$ [height can't be negative]

$\therefore$ $y=\sqrt{2k}$ is the turning point.

Since, $\frac{dV}{dy}>0$, therefore, Volume is maximum at $y=\sqrt{2k}$

$\therefore$ From eq. (i), $x^2=\frac{k^2}{2k+2k}=\frac{k^2}{4k}=\frac{k}{4}$

$\Rightarrow \quad x = \frac{\sqrt{k}}{2}$

Now Semi-vertical angle of the cone $\sin\theta = \frac{x}{\sqrt{x^2+y^2}} = \frac{\frac{\sqrt{k}}{2}}{\sqrt{\frac{k}{4}+2k}} = \frac{\sqrt{k}}{2} \times \sqrt{\frac{4}{9k}} = \frac{1}{3}$

$\Rightarrow \quad \theta = \sin^{-1}\frac{1}{3}$

Choose the correct answer in the Exercises 27 to 29.

27. The point on the curve $x^2 = 2y$ which is nearest to the point (0, 5) is:

(A) $\left(2\sqrt{2}, 4\right)$

(B) $\left(2\sqrt{2}, 0\right)$

(C) (0, 0)

(D) (2, 2)

Ans. Equation of the curve is $x^2 = 2y$ ....(i)

Let P$\left(x, y\right)$ be any point on the curve (i), then according to question,

Distance between given point (0, 5) and P = $\sqrt{\left(x-2\right)^2+\left(y-5\right)^2} = z$ (say)

$\Rightarrow \quad z^2 = x^2 + \left(y-5\right)^2$

= $2y + \left(y-5\right)^2$ [From eq. (i)]

$\Rightarrow \quad z^2 = 2y + y^2 + 25 - 10y$

$\Rightarrow z^2 = y^2 - 8y + 25$ = Z (say)

$\Rightarrow \frac{dZ}{dy} = 2y - 8$ and $\frac{d^2Z}{dy^2} = 2$

Now $\frac{dZ}{dy} = 0$

$\Rightarrow 2y - 8 = 0$

$\Rightarrow y = 4$

At $y = 4$

$\frac{d^2Z}{dy^2} = 2$ [Positive]

$\therefore$ Z is minimum and z is minimum at $y = 4$

$\therefore$ From eq. (i)

$x^2 = 8$

$\Rightarrow x = \pm 2\sqrt{2}$

$\therefore \left(2\sqrt{2}, 4\right)$ and $\left(-2\sqrt{2}, 4\right)$ are two points on curve (i) which are nearest to (0, 5).

Therefore, option (A) is correct.

28. For all real values of x, the minimum value of $\frac{1-x+x^2}{1+x+x^2}$ is:

(A) 0

(B) 1

(C) 3

(D) $\frac{1}{3}$

Ans. Given: $f(x)=\frac{1-x+x^2}{1+x+x^2}$ ....(i)

$$\Rightarrow f'(x)=\frac{(1+x+x^2)\frac{d}{dx}(1-x+x^2)-(1-x+x^2)\frac{d}{dx}(1+x+x^2)}{(1+x+x^2)^2}$$

$$\Rightarrow f'(x)=\frac{(1+x+x^2)(-1+2x)-(1-x+x^2)(1+2x)}{(1+x+x^2)^2}$$

$$\Rightarrow f'(x)=\frac{-1+2x-x+2x^2-x^2+2x^3-1-2x+x+2x^2-x^2-2x^3}{(1+x+x^2)^2}$$

$$\Rightarrow f'(x)=\frac{-2+2x^2}{(1+x+x^2)^2}=\frac{-2(1-x^2)}{(1+x+x^2)^2}$$

Now $f'(x)=0$

$$\Rightarrow \frac{-2(1-x^2)}{(1+x+x^2)^2}=0$$

$$\Rightarrow -2(1-x^2)=0$$

$$\Rightarrow 1-x^2=0$$

$$\Rightarrow x^2=1$$

$$\Rightarrow x=\pm 1$$

$\therefore$ $x=1$ and $x=-1$ [Turning points]

At $x=-1$,

from eq. (i),

$$f(-1)=\frac{1+1+1}{1-1+1}=3$$

At $x=1$,

from eq. (i),

$f(1)=\frac{1-1+1}{1+1+1}=\frac{1}{3}$ [Minimum value]

Therefore, option (D) is correct.

---

29. The maximum value of $\left[x(x-1)+1\right]^{1/3}$, $0\le x\le 1$ is:

(A) $\left(\frac{1}{3}\right)^{1/3}$

(B) $\frac{1}{2}$

(C) 1

(D) $\frac{1}{3}$

Ans. Let $f(x)=\left[x(x-1)+1\right]^{\frac{1}{3}}$

= $\left(x^2-x+1\right)^{\frac{1}{3}}$, $0\le x\le 1$ ....(i)

$$\therefore\ f'(x)=\frac{1}{3}\left(x^2-x+1\right)^{\frac{-2}{3}}\frac{d}{dx}\left(x^2-x+1\right)$$

= $\dfrac{(2x-1)}{3(x^2-x+1)^{\frac{2}{3}}}$

Now $f'(x)=0$

$\Rightarrow \dfrac{(2x-1)}{3(x^2-x+1)^{\frac{2}{3}}}$ = 0

$\Rightarrow 2x-1=0$

$\Rightarrow x=\dfrac{1}{2}$ [Turning point] and it belongs to the given enclosed interval $0\le x\le 1$ i.e., [0, 1].

At $x=\dfrac{1}{2}$, from eq. (i),

$$f\left(\frac{1}{2}\right)=\left(\frac{1}{4}-\frac{1}{2}+1\right)^{\frac{1}{3}}=\left(\frac{1-2+4}{4}\right)^{\frac{1}{3}}=\left(\frac{3}{4}\right)^{\frac{1}{3}}<1$$

At $x=0$, from eq. (i),

$$f(0)=(1)^{\frac{1}{3}}=1$$

At $x=1$, from eq. (i),

$$f(1)=(1-1+1)^{\frac{1}{3}}=(1)^{\frac{1}{3}}=1$$

$\therefore$ Maximum value of $f(x)$ is 1.

Therefore, option (C) is correct.