

CBSE Class-12 Mathematics

NCERT solution

Chapter - 5

Continuity & Differentiability - Miscellaneous Exercise

Differentiate with respect to  $x$  the functions in Exercises 1 to 5.

1.  $(3x^2 - 9x + 5)^9$

Ans. Let  $y = (3x^2 - 9x + 5)^9$

$$\therefore \frac{dy}{dx} = 9(3x^2 - 9x + 5)^8 \frac{d}{dx}(3x^2 - 9x + 5)$$

$$\left[ \because \frac{d}{dx} \{f(x)\}^n = n \{f(x)\}^{n-1} \frac{d}{dx} f(x) \right]$$

$$\Rightarrow \frac{dy}{dx} = 9(3x^2 - 9x + 5)^8 [3(2x) - 9(1) + 0] = 9(3x^2 - 9x + 5)^8 [6x - 9]$$

$$\Rightarrow \frac{dy}{dx} = 27(3x^2 - 9x + 5)^8 [2x - 3]$$

2.  $\sin^3 x + \cos^6 x$

Ans. Let  $y = \sin^3 x + \cos^6 x = (\sin x)^3 + (\cos x)^6$

$$\therefore \frac{dy}{dx} = 3(\sin x)^2 \frac{d}{dx} \sin x + 6(\cos x)^5 \frac{d}{dx} \cos x$$

$$\Rightarrow \frac{dy}{dx} = 3 \sin^2 x \cos x - 6 \cos^5 x \sin x$$

$$= 3 \sin x \cos x (\sin x - 2 \cos^4 x)$$

3.  $(5x)^{3\cos 2x}$

Ans. Let  $y = (5x)^{3\cos 2x}$  .....(i)

$$\Rightarrow \log y = \log (5x)^{3\cos 2x}$$

$$\Rightarrow \log y = 3 \cos 2x \log (5x)$$

$$\therefore \frac{d}{dx} \log y = 3 \frac{d}{dx} \{ \cos 2x \log (5x) \}$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = 3 \left[ \cos 2x \frac{d}{dx} \log (5x) + \log (5x) \frac{d}{dx} \cos 2x \right]$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = 3 \left[ \cos 2x \frac{1}{5x} \frac{d}{dx} 5x + \log (5x) (-\sin 2x) \frac{d}{dx} 2x \right]$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = 3 \left[ \cos 2x \frac{1}{5x} \cdot 5 - 2 \sin 2x \log (5x) \right]$$

$$\Rightarrow \frac{dy}{dx} = 3y \left[ \frac{\cos 2x}{x} - 2 \sin 2x \log (5x) \right]$$

$$\Rightarrow \frac{dy}{dx} = 3(5x)^{3\cos 2x} \left[ \frac{\cos 2x}{x} - 2 \sin 2x \log (5x) \right]$$

4.  $\sin^{-1}(x\sqrt{x}), 0 \leq x \leq 1$

Ans. Let  $y = \sin^{-1}(x\sqrt{x}) = \sin^{-1}\left(x^{\frac{3}{2}}\right)$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1 - \left(x^{\frac{3}{2}}\right)^2}} \frac{d}{dx} x^{\frac{3}{2}}$$

$$= \frac{1}{\sqrt{1-x^3}} \cdot \frac{3}{2} x^{\frac{1}{2}}$$

$$= \frac{3\sqrt{x}}{2\sqrt{1-x^3}}$$

$$= \frac{3}{2} \sqrt{\frac{x}{1-x^3}}$$

5.  $\frac{\cos^{-1} \frac{x}{2}}{\sqrt{2x+7}}, -2 < x < 2$

Ans. Let  $y = \frac{\cos^{-1} \frac{x}{2}}{\sqrt{2x+7}}$

$$\therefore \frac{dy}{dx} = \frac{\sqrt{2x+7} \frac{d}{dx} \cos^{-1} \frac{x}{2} - \cos^{-1} \frac{x}{2} \frac{d}{dx} \sqrt{2x+7}}{(\sqrt{2x+7})^2} \quad [\text{By Quotient Rule}]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{2x+7} \left( \frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \right) \frac{d}{dx} \frac{x}{2} - \left( \cos^{-1} \frac{x}{2} \right) \frac{1}{2} (2x+7)^{-\frac{1}{2}} \frac{d}{dx} (2x+7)}{(\sqrt{2x+7})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sqrt{2x+7} \cdot \frac{2}{\sqrt{4-x^2}} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{\cos^{-1} \frac{x}{2}}{\sqrt{2x+7}} \times 2}{(2x+7)}$$

$$= \frac{- \left[ \frac{\sqrt{2x+7}}{\sqrt{4-x^2}} + \frac{\cos^{-1} \frac{x}{2}}{\sqrt{2x+7}} \right]}{(2x+7)}$$

$$\Rightarrow \frac{dy}{dx} = - \left[ \frac{2x+7 + \sqrt{4-x^2} \cos^{-1} \frac{x}{2}}{\sqrt{4-x^2} \sqrt{2x+7} (2x+7)} \right]$$

$$= - \left[ \frac{2x+7 + \sqrt{4-x^2} \cos^{-1} \frac{x}{2}}{\sqrt{4-x^2} (2x+7)^{\frac{3}{2}}} \right]$$

$$= - \left[ \frac{1}{\sqrt{4-x^2} \sqrt{2x+7}} + \frac{\cos^{-1} \frac{x}{2}}{(2x+7)^{\frac{3}{2}}} \right]$$

**Differentiate with respect to  $x$  the functions in Exercises 6 to 11.**

6.  $\cot^{-1} \left[ \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right], 0 < x < \frac{\pi}{2}$

**Ans.** Let  $y = \cot^{-1} \left( \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right), 0 < x < \frac{\pi}{2}$  .....(i)

Now,  $\sqrt{1+\sin x} = \sqrt{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}}$

$$= \sqrt{\left( \cos \frac{x}{2} + \sin \frac{x}{2} \right)^2} = \cos \frac{x}{2} + \sin \frac{x}{2}$$

$$\text{And } \sqrt{1 - \sin x} = \sqrt{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} - 2 \sin \frac{x}{2} \cos \frac{x}{2}}$$

$$= \sqrt{\left(\cos \frac{x}{2} - \sin \frac{x}{2}\right)^2} = \cos \frac{x}{2} - \sin \frac{x}{2}$$

$$\therefore \text{ From eq. (i), } y = \cot^{-1} \left( \frac{\cos \frac{x}{2} + \sin \frac{x}{2} + \cos \frac{x}{2} - \sin \frac{x}{2}}{\cos \frac{x}{2} + \sin \frac{x}{2} - \cos \frac{x}{2} + \sin \frac{x}{2}} \right)$$

$$= \cot^{-1} \left( \frac{2 \cos \frac{x}{2}}{2 \sin \frac{x}{2}} \right)$$

$$\Rightarrow y = \cot^{-1} \left( \cot \frac{x}{2} \right) = \frac{x}{2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2} (1) = \frac{1}{2}$$

7.  $(\log x)^{\log x}, x > 1$

**Ans.** Let  $y = (\log x)^{\log x}, x > 1$  .....(i)

Taking log both sides, we get

$$\Rightarrow \log y = \log (\log x)^{\log x} = \log x \log (\log x)$$

$$\therefore \frac{d}{dx} (\log y) = \frac{d}{dx} (\log x \log (\log x))$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \log x \frac{d}{dx} \log (\log x) + \log (\log x) \frac{d}{dx} \log x$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \log x \cdot \frac{1}{\log x} \frac{d}{dx} (\log x) + \log(\log x) \frac{1}{x}$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \log x \cdot \frac{1}{\log x} \cdot \frac{1}{x} + \log(\log x) \frac{1}{x}$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{1}{x} + \log(\log x) \frac{1}{x}$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \left[ \frac{1 + \log(\log x)}{x} \right]$$

$$\Rightarrow \frac{dy}{dx} = y \left( \frac{1 + \log(\log x)}{x} \right)$$

$$= (\log x)^{\log x} \left( \frac{1 + \log(\log x)}{x} \right)$$

8.  $\cos(a \cos x + b \sin x)$  for some constants  $a$  and  $b$ .

**Ans.** Let  $y = \cos(a \cos x + b \sin x)$  for some constants  $a$  and  $b$

$$\therefore \frac{dy}{dx} = -\sin(a \cos x + b \sin x) \frac{d}{dx} (a \cos x + b \sin x)$$

$$\Rightarrow \frac{dy}{dx} = -\sin(a \cos x + b \sin x) (-a \sin x + b \cos x)$$

$$\Rightarrow \frac{dy}{dx} = -(-a \sin x + b \cos x) \sin(a \cos x + b \sin x)$$

$$\Rightarrow \frac{dy}{dx} = (a \sin x - b \cos x) \sin(a \cos x + b \sin x)$$

9.  $(\sin x - \cos x)^{\sin x - \cos x}, \frac{\pi}{4} < x < \frac{3\pi}{4}$

**Ans.** Let  $y = (\sin x - \cos x)^{\sin x - \cos x}$  .....(i)

Taking log Both Sides, we get

$$\Rightarrow \log y = \log (\sin x - \cos x)^{\sin x - \cos x}$$

$$\Rightarrow \log y = (\sin x - \cos x) \log (\sin x - \cos x)$$

$\Rightarrow$

$$\frac{d}{dx} \log y = (\sin x - \cos x) \frac{d}{dx} \log (\sin x - \cos x) + \log (\sin x - \cos x) \frac{d}{dx} (\sin x - \cos x)$$

$\Rightarrow$

$$\frac{1}{y} \frac{dy}{dx} = (\sin x - \cos x) \frac{1}{(\sin x - \cos x)} \frac{d}{dx} (\sin x - \cos x) + \log (\sin x - \cos x) \cdot (\cos x + \sin x)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = (\cos x + \sin x) + (\cos x + \sin x) \log (\sin x - \cos x)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = (\cos x + \sin x) [1 + \log (\sin x - \cos x)]$$

$$\Rightarrow \frac{dy}{dx} = y (\cos x + \sin x) [1 + \log (\sin x - \cos x)]$$

$$\Rightarrow \frac{dy}{dx} = (\sin x - \cos x)^{\sin x - \cos x} (\cos x + \sin x) [1 + \log (\sin x - \cos x)]$$

**10.  $x^x + x^a + a^x + a^a$ , for some fixed  $a > 0$  and  $x > 0$ .**

**Ans.** Let  $y = x^x + x^a + a^x + a^a$

$$\therefore \frac{dy}{dx} = \frac{d}{dx} x^x + \frac{d}{dx} x^a + \frac{d}{dx} a^x + \frac{d}{dx} a^a$$

$$= \frac{d}{dx} x^x + ax^{a-1} + a^x \log a + 0 \quad \dots\dots(i)$$

Now taking  $\frac{d}{dx} (x^x)$ , let  $u = x^x \quad \dots\dots(ii)$

Taking log both sides, we get

$$\Rightarrow \log u = \log x^x = x \log x$$

$$\Rightarrow \frac{d}{dx} \log u = \frac{d}{dx} (x \log x)$$

$$\Rightarrow \frac{1}{u} \frac{du}{dx} = x \frac{d}{dx} \log x + \log x \frac{d}{dx} x$$

$$\Rightarrow \frac{1}{u} \frac{du}{dx} = x \frac{1}{x} + \log x \cdot 1 = 1 + \log x$$

$$\Rightarrow \frac{du}{dx} = u(1 + \log x)$$

$\therefore$  From eq. (ii),

$$\frac{d}{dx} x^x = x^x (1 + \log x)$$

$\therefore$  From eq. (i),

$$\Rightarrow \frac{dy}{dx} = x^x (1 + \log x) + ax^{a-1} + a^x \log a$$

**11.**  $x^{x^2-3} + (x-3)^{x^2}$  for  $x > 3$ .

**Ans.** Let  $y = x^{x^2-3} + (x-3)^{x^2}$  for  $x > 3$

Putting  $u = x^{x^2-3}$  and  $v = (x-3)^{x^2}$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \dots\dots\dots(i)$$

Now  $u = x^{x^2-3}$

$$\Rightarrow \log u = \log x^{x^2-3} = (x^2 - 3) \log x$$



$$\begin{aligned}\therefore \frac{1}{u} \frac{du}{dx} &= (x^2 - 3) \frac{d}{dx} \log x + \log x \frac{d}{dx} (x^2 - 3) \\&= (x^2 - 3) \frac{1}{x} + \log x (2x - 0) \\&\Rightarrow \frac{1}{u} \frac{du}{dx} = \frac{x^2 - 3}{x} + 2x \log x \\&\Rightarrow \frac{du}{dx} = u \left( \frac{x^2 - 3}{x} + 2x \log x \right) \\&\Rightarrow \frac{du}{dx} = x^{x^2-3} \left( \frac{x^2 - 3}{x} + 2x \log x \right) \dots\dots\dots(ii)\end{aligned}$$

Again  $v = (x-3)^{x^2}$

$$\begin{aligned}&\Rightarrow \log v = \log (x-3)^{x^2} = x^2 \log (x-3) \\&\therefore \frac{1}{v} \frac{dv}{dx} = x^2 \frac{d}{dx} \log (x-3) + \log (x-3) \frac{d}{dx} x^2 \\&= x^2 \frac{1}{x-3} \frac{d}{dx} (x-3) + \log (x-3) 2x \\&\Rightarrow \frac{1}{v} \frac{dv}{dx} = \frac{x^2}{x-3} + 2x \log (x-3) \\&\Rightarrow \frac{dv}{dx} = v \left[ \frac{x^2}{x-3} + 2x \log (x-3) \right] \\&\Rightarrow \frac{dv}{dx} = (x-3)^{x^2} \left[ \frac{x^2}{x-3} + 2x \log (x-3) \right] \dots\dots\dots(iii)\end{aligned}$$

Putting the values from eq. (ii) and (iii) in eq. (i),

$$\frac{dy}{dx} = x^{x^2-3} \left( \frac{x^2-3}{x} + 2x \log x \right) + (x-3)^{x^2} \left[ \frac{x^2}{x-3} + 2x \log (x-3) \right]$$

**12. Find  $\frac{dy}{dx}$  if  $y = 12(1 - \cos t)$  and  $x = 10(t - \sin t)$ ,  $-\frac{\pi}{2} < t < \frac{\pi}{2}$ .**

**Ans. Given:**  $y = 12(1 - \cos t)$  and  $x = 10(t - \sin t)$

$$\therefore \frac{dy}{dt} = 12 \frac{d}{dt}(1 - \cos t) = 12(0 + \sin t) = 12 \sin t$$

$$\text{and } \frac{dx}{dt} = 10 \frac{d}{dt}(t - \sin t)$$

$$\Rightarrow \frac{dx}{dt} = 10(1 - \cos t)$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12 \sin t}{10(1 - \cos t)}$$

$$= \frac{6}{5} \cdot \frac{2 \sin \frac{t}{2} \cos \frac{t}{2}}{2 \sin^2 \frac{t}{2}}$$

$$= \frac{6}{5} \cdot \frac{\cos \frac{t}{2}}{\sin \frac{t}{2}}$$

$$= \frac{6}{5} \cot \frac{t}{2}$$

**13. Find  $\frac{dy}{dx}$  if  $y = \sin^{-1} x + \sin^{-1} \sqrt{1-x^2}$ ,  $-1 \leq x \leq 1$ .**

**Ans. Given:**  $y = \sin^{-1} x + \sin^{-1} \sqrt{1-x^2}$

$$\begin{aligned}
 \therefore \frac{dy}{dx} &= \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}} \frac{d}{dx} \sqrt{1-x^2} \\
 \Rightarrow \frac{dy}{dx} &= \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}} \frac{1}{2} (1-x^2)^{-\frac{1}{2}} \frac{d}{dx} (1-x^2) \\
 \Rightarrow \frac{dy}{dx} &= \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-1+x^2}} \frac{1}{2\sqrt{1-x^2}} (-2x) \\
 &= \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{x^2}} \left( \frac{-x}{\sqrt{1-x^2}} \right) \\
 \Rightarrow \frac{dy}{dx} &= \frac{1}{\sqrt{1-x^2}} - \frac{x}{x\sqrt{1-x^2}} \\
 &= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \\
 &= 0
 \end{aligned}$$

14. If  $x\sqrt{1+y} + y\sqrt{1+x} = 0$ , for  $-1 < x < 1$ , prove that  $\frac{dy}{dx} = \frac{-1}{(1+x)^2}$ .

**Ans.** Given:  $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$$\Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$$

Squaring both sides, We get

$$\Rightarrow x^2(1+y) = y^2(1+x)$$

$$\Rightarrow x^2 + x^2y = y^2 + y^2x$$

$$\Rightarrow x^2 - y^2 = -x^2y + y^2x$$

$$\Rightarrow (x-y)(x+y) = -xy(x-y)$$

$$\Rightarrow x+y = -xy$$

$$\Rightarrow y+xy = -x$$

$$\Rightarrow y(1+x) = -x$$

$$\Rightarrow y = \frac{-x}{1+x}$$

$$\therefore \frac{dy}{dx} = \frac{(1+x) \frac{d}{dx}(-x) - (-x) \frac{d}{dx}(1+x)}{(1+x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(1+x)+x}{(1+x)^2}$$

$$= -\frac{1}{(1+x)^2} \quad \text{Proved.}$$

15. If  $(x-a)^2 + (y-b)^2 = c^2$ , for some  $c > 0$ , prove that  $\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$  is a constant

independent of a and b.

**Ans.** Given:  $(x-a)^2 + (y-b)^2 = c^2$ , .....(i)

$$\therefore 2(x-a) + 2(y-b) \frac{dy}{dx} = 0$$

$$\Rightarrow 2(x-a) = -2(y-b) \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{x-a}{y-b}\right) \text{ .....(ii)}$$

$$\begin{aligned} \text{Again } \frac{d^2y}{dx^2} &= \frac{-\left[(y-b) \cdot 1 - (x-a) \frac{dy}{dx}\right]}{(y-b)^2} \\ \Rightarrow \frac{d^2y}{dx^2} &= \frac{-\left[(y-b) \cdot 1 - (x-a) \left(\frac{-(x-a)}{y-b}\right)\right]}{(y-b)^2} \quad [\text{From eq. (ii)}] \\ \Rightarrow \frac{d^2y}{dx^2} &= \frac{-\left[(y-b) + \frac{(x-a)^2}{y-b}\right]}{(y-b)^2} \\ &= \frac{-\left[(y-b)^2 + (x-a)^2\right]}{(y-b)^3} \\ &= \frac{-c^2}{(y-b)^3} \dots\dots\dots(\text{iii}) \end{aligned}$$

Putting values of  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  in the given expression,

$$\begin{aligned} &\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \\ &= \frac{\left[1 + \frac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{\frac{-c^2}{(y-b)^3}} \end{aligned}$$

$$= \frac{\left[(y-b)^2 + (x-a)^2\right]^{\frac{3}{2}}}{(y-b)^3} \times \frac{(y-b)^3}{-c^2} = \frac{(c^2)^{\frac{3}{2}}}{-c^2} = -c$$

which is a constant and is independent of  $a$  and  $b$ .

**16. If  $\cos y = x \cos(a+y)$  with  $\cos a \neq \pm 1$ , prove that  $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$ .**

**Ans.** Given:  $\cos y = x \cos(a+y)$

$$\Rightarrow x = \frac{\cos y}{\cos(a+y)}$$

$$\therefore \frac{dx}{dy} = \frac{d}{dy} \left( \frac{\cos y}{\cos(a+y)} \right)$$

$$\Rightarrow \frac{dx}{dy} = \frac{\cos(a+y) \frac{d}{dy} \cos y - \cos y \frac{d}{dy} \cos(a+y)}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dx}{dy} = \frac{\cos(a+y)(-\sin y) - \cos y \{-\sin(a+y)\}}{\cos^2(a+y)}$$

$$= \frac{-\cos(a+y) \sin y + \sin(a+y) \cos y}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dx}{dy} = \frac{\sin(a+y-y)}{\cos^2(a+y)}$$

$$= \frac{\sin a}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a} \quad [\text{Taking reciprocal}]$$

17. If  $x = a(\cos t + t \sin t)$  and  $y = a(\sin t - t \cos t)$ , find  $\frac{d^2y}{dx^2}$ .

**Ans.** Given:  $x = a(\cos t + t \sin t)$  and  $y = a(\sin t - t \cos t)$

Differentiating both sides with respect to  $t$ ,

$$\frac{dx}{dt} = a \left( -\sin t + \frac{d}{dt} t \sin t \right)$$

$$\text{and } \frac{dy}{dt} = a \left( \cos t - \frac{d}{dt} t \cos t \right)$$

$$\Rightarrow \frac{dx}{dt} = a \left( -\sin t + t \frac{d}{dt} \sin t + \sin t \frac{d}{dt} t \right)$$

$$\text{and } \frac{dy}{dt} = a \left( \cos t - \left( t \frac{d}{dt} \cos t + \cos t \frac{d}{dt} t \right) \right)$$

$$\Rightarrow \frac{dx}{dt} = a(-\sin t + t \cos t + \sin t)$$

$$\text{and } \frac{dy}{dt} = a(\cos t - (-t \sin t + \cos t))$$

$$\Rightarrow \frac{dx}{dt} = at \cos t \quad \text{and} \quad \frac{dy}{dt} = at \sin t$$

$$\text{Now } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{at \sin t}{at \cos t} = \frac{\sin t}{\cos t} = \tan t$$

$$\text{Again } \frac{d^2y}{dx^2} = \frac{d}{dx} \tan t = \sec^2 t \frac{d}{dx} t$$

$$= \sec^2 t \frac{dt}{dx} = \sec^2 t \frac{1}{at \cos t}$$

$$= \sec^2 t \frac{\sec t}{at} = \frac{\sec^3 t}{at}$$

**18. If  $f(x) = |x|^3$ , show that  $f''(x)$  exists for all real  $x$  and find it.**

**Ans.** Given:  $f(x) = |x|^3 = \begin{cases} x^3, & \text{if } x \geq 0 \\ (-x^3), & \text{if } x < 0 \end{cases}$

Now, L.H.D. at  $x = 0$

$$\Rightarrow \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \rightarrow 0^-} \frac{-x^3 - 0}{x - 0}$$

$$= \lim_{x \rightarrow 0^-} -x^2 = 0$$

$\therefore$  L.H.D. at  $x = 0$  = R.H.D. at  $x = 0$

Therefore,  $f(x)$  is differentiable at  $x = 0$ .

$$\therefore f'(x) = \begin{cases} 3x^2, & \text{if } x \geq 0 \\ -3x^2, & \text{if } x < 0 \end{cases}$$

Now, L.H.D. at  $x = 0$

$$\Rightarrow \lim_{x \rightarrow 0^-} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \rightarrow 0^-} \frac{-3x^2 - 0}{x - 0} = \lim_{x \rightarrow 0^-} (-3x) = 0$$

And R.H.D. at  $x = 0$



$$\Rightarrow \lim_{x \rightarrow 0^+} \frac{f'(x) - f'(0)}{x - 0}$$

$$= \lim_{x \rightarrow 0^+} \frac{-3x^2 - 0}{x - 0}$$

$$\lim_{x \rightarrow 0^+} -3x = 0$$

$\therefore$  Again L.H.D. at  $x = 0$  = R.H.D. at  $x = 0$

**19. Using mathematical induction, prove that  $\frac{d}{dx}(x^n) = nx^{n-1}$  for all positive integers  $n$ .**

**Ans.** Let  $p(n)$  be the given statement in the problem.

$$\Rightarrow p(n) = \frac{d}{dx}(x^n) = nx^{n-1} \dots(i)$$

$$\therefore p(1) = \frac{d}{dx}(x^1) = (1)x^{1-1} = (1)x^0 = 1,$$

which is true as  $\frac{d}{dx}(x) = 1$

Now we suppose  $p(m)$  is true.

$$\therefore p(m) = \frac{d}{dx}(x^m) = mx^{m-1} \dots(ii)$$

To establish the truth of  $p(m+1)$ , we prove,

$$p(m+1) = \frac{d}{dx}(x^{m+1}) = (m+1)x^{m+1-1}$$

$$\Rightarrow x^{m+1} = x^1 + x^m$$

$$\Rightarrow \frac{d}{dx} x^{m+1} = \frac{d}{dx} (x \cdot x^m)$$

$$= x \cdot \frac{d}{dx} x^m + x^m \frac{d}{dx} x$$

$$= x \cdot m x^{m-1} + x^m (1)$$

$$\therefore m x^m + x^m = x^m (m+1)$$

$$\Rightarrow (m+1) x^m = (m+1) x^m$$

$$\Rightarrow (m+1) x^{(m+1)-1}$$

Therefore,  $p(m+1)$  is true if  $p(m)$  is true but  $p(1)$  is true.

$\therefore$  By Principal of Induction  $p(n)$  is true for all  $n \in \mathbb{N}$ .

**20. Using the fact that  $\sin(A+B) = \sin A \cos B + \cos A \sin B$  and the differentiation, obtain the sum formula for cosines.**

**Ans.** Given:  $\sin(A+B) = \sin A \cos B + \cos A \sin B$

Consider A and B as function of  $t$  and differentiating both sides w.r.t.  $x$ ,

$$\cos(A+B) \left( \frac{dA}{dt} + \frac{dB}{dt} \right) = \sin A (-\sin B) \frac{dB}{dt} + \cos B \left( \cos A \frac{dA}{dt} \right) + \cos A \cos B \frac{dB}{dt} + \sin B (-\sin A) \frac{dA}{dt}$$

$$\Rightarrow \cos(A+B) \left( \frac{dA}{dt} + \frac{dB}{dt} \right) = (\cos A \cos B - \sin A \sin B) \left( \frac{dA}{dt} + \frac{dB}{dt} \right)$$

$$\Rightarrow \cos(A+B) = (\cos A \cos B - \sin A \sin B)$$

**21. Does there exist a function which is continuous everywhere but not differentiable at**

exactly two points?

**Ans.** Let us consider the function  $f(x) = |x| + |x-1|$

$f$  is continuous everywhere but it is not differentiable at  $x=0$  and  $x=1$ .

22. If  $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$ , prove that  $\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$ .

**Ans.** Given:  $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

$\therefore$

$$\begin{aligned} \frac{dy}{dx} &= \begin{vmatrix} \frac{d}{dx}f(x) & \frac{d}{dx}g(x) & \frac{d}{dx}h(x) \\ l & m & n \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ 0 & 0 & 0 \\ a & b & c \end{vmatrix} + y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ 0 & 0 & 0 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 & 1 \\ f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix} \end{aligned}$$

23. If  $y = e^{a \cos^{-1} x}$ ,  $-1 \leq x \leq 1$ , show that  $(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0$ .

**Ans.** Given:  $y = e^{a \cos^{-1} x}$

$$\therefore \frac{dy}{dx} = e^{a \cos^{-1} x} \cdot \frac{d}{dx} a \cos^{-1} x$$

$$= e^{a \cos^{-1} x} \cdot a \left( \frac{-1}{\sqrt{1-x^2}} \right)$$

$$= \frac{-ay}{\sqrt{1-x^2}}$$

$$\Rightarrow \left( \frac{dy}{dx} \right)^2 = \frac{a^2 y^2}{1-x^2}$$

$$\Rightarrow (1-x^2) \left( \frac{dy}{dx} \right)^2 = a^2 y^2$$

Differentiating both sides with respect to  $x$ ,

$$(1-x^2) 2 \cdot \frac{dy}{dx} \cdot \frac{d^2 y}{dx^2} + \left( \frac{dy}{dx} \right)^2 (-2x) = 2a^2 y \frac{dy}{dx}$$

$$\Rightarrow (1-x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} = a^2 y$$

$$\Rightarrow (1-x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} - a^2 y = 0 \quad \text{Proved.}$$