CBSE Class–12 Mathematics

NCERT solution

Chapter - 6

Application of Derivatives - Miscellaneous Exercise

1. Using differentials, find the approximate value of each of the following:

(a) $\left(\frac{17}{81}\right)^{1/4}$

(b) $(33)^{-1/5}$

Ans. (a) $\left(\frac{17}{81}\right)^{\frac{1}{4}}$

Let $y = x^{\frac{1}{4}}$ ....(i)

$$\Rightarrow \frac{dy}{dx} = \frac{1}{4x^{\frac{3}{4}}}$$

$$\Rightarrow dy = \frac{dx}{4\left(x^{\frac{1}{4}}\right)^3}$$

$$= \frac{\Delta x}{4\left(x^{\frac{1}{4}}\right)^3} \text{ ....(ii)}$$

Changing x to $x + \Delta x$ and y to $y + \Delta$ in eq. (i), we have

$$y + \Delta y = (x + \Delta x)^{\frac{1}{4}} = \left(\frac{17}{81}\right)^{\frac{1}{4}} = \left(\frac{16}{81} + \frac{1}{81}\right)^{\frac{1}{4}} \text{ ....(iii)}$$

Here $x=\frac{16}{81}$ and $\Delta x=\frac{1}{81}$

$$\therefore\ x^{\frac{1}{4}}=\left(\frac{16}{81}\right)^{\frac{1}{4}}=\frac{2}{3}$$

From eq. (iii),

$$\left(\frac{17}{81}\right)^{\frac{1}{4}}=y+\Delta y\sim y+\ dy\sim x^{\frac{1}{4}}+\frac{\Delta x}{4\left(x^{\frac{1}{4}}\right)^{3}}$$

$$\Rightarrow\left(\frac{17}{81}\right)^{\frac{1}{4}}\sim\frac{2}{3}+\frac{\frac{1}{81}}{4\left(\frac{2}{3}\right)^{3}}\sim\frac{2}{3}+\frac{1}{81}\times\frac{27}{32}\sim\frac{2}{3}+\frac{1}{96}=\frac{65}{96}=0.677$$

(b) $(33)^{\frac{-1}{5}}$

Let $y=x^{\frac{-1}{5}}$ ....(i)

$$\Rightarrow\frac{dy}{dx}=\frac{-1}{5x^{\frac{6}{5}}}$$

$$\Rightarrow\ dy=\frac{-dx}{5\left(x^{\frac{1}{5}}\right)^{6}}$$

$$=\frac{-\Delta x}{5\left(x^{\frac{1}{5}}\right)^{6}}\ ....(ii)$$

Changing x to $x+\Delta x$ and y to $y+\Delta$ in eq. (i), we have

$y+\Delta y=(x+\Delta x)^{\frac{-1}{5}}=(33)^{\frac{-1}{5}}=(32+1)^{\frac{-1}{5}}$ ....(iii)

Here $x=32$ and $\Delta x=1$

$\therefore x^{\frac{-1}{5}}=(32)^{\frac{-1}{5}}=\frac{1}{2}$

From eq. (iii),

$$(33)^{\frac{-1}{5}}=y+\Delta y\sim y+dy\sim x^{\frac{-1}{5}}+\frac{\Delta x}{5\left(x^{\frac{1}{5}}\right)^6}$$

$$\Rightarrow (33)^{-1/5}\sim\frac{1}{2}-\frac{1}{5(2)^6}\sim\frac{1}{2}-\frac{1}{5}\times\frac{1}{64}\sim\frac{1}{2}-\frac{1}{320}=\frac{159}{320}=0.497$$

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2. Show that the function given by $f(x)=\frac{\log x}{x}$ has maximum value at $x=e$.

Ans. Here $f(x)=\frac{\log x}{x}$, $x>0$ ....(i)

$\therefore f'(x)=\frac{x.\frac{1}{x}-\log x.1}{x^2}=\frac{1-\log x}{x^2}$ ....(ii)

And $f''(x)=\frac{x^2\left(\frac{-1}{x}\right)-(1-\log x).2x}{x^4}=\frac{-x-2x+2x\log x}{x^4}$

$=\frac{2x\log x-3x}{x^4}$

$\Rightarrow f''(x)=\frac{x(2\log x-3)}{x^4}=\frac{2\log x-3}{x^3}$ ....(iii)

Now $f'(x)=0$

$\Rightarrow \frac{1-\log x}{x^2}=0$

$\Rightarrow 1-\log x=0$

$\Rightarrow \log x=1$

$\Rightarrow \log x=\log e$

$\Rightarrow x=e$

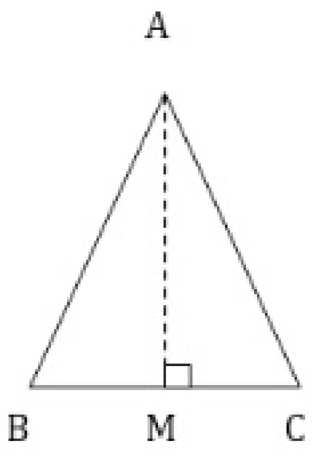
$\therefore$ From eq. (iii),

$f''(x)=\frac{2\log e-3}{e^3}=\frac{2-3}{e^3}=\frac{-1}{e^3}<0$

$\therefore$ $x=e$ is a point of local maxima and maximum value of $f(x)$ is at $x=e$.

3. The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base?

Ans. Let BC = b be the fixed base and AB = AC = x be
the two equal sides of given isosceles triangle.



Since $\frac{dx}{dt} = -3$ cm/s ....(i)

Area of $\Delta ABC = \frac{1}{2}$ x BC x AM

= $\frac{b}{2}\sqrt{x^2 - \frac{b^2}{4}}$

= $\frac{b}{2}\sqrt{\frac{4x^2 - b^2}{4}} = \frac{b}{4}\sqrt{4x^2 - b^2}$

$\therefore \frac{d\Delta}{dt} = \frac{d}{dt}\left(\frac{b}{4}\sqrt{4x^2 - b^2}\right)$

= $\frac{b}{4} \times \frac{d}{dx}\left(\sqrt{4x^2 - b^2}\right) \times \frac{dx}{dt}$ [By chain rule]

$\Rightarrow \frac{d\Delta}{dt} = \frac{b}{4} \times \frac{1}{2\sqrt{4x^2 - b^2}} \times 8x \times (-3)$

= $\frac{-3bx}{\sqrt{4x^2 - b^2}}$ cm^2/s

Now when $x = b$,

$\frac{d\Delta}{dt} = \frac{-3b.b}{\sqrt{4b^2 - b^2}} = \frac{-3b^2}{\sqrt{3}b} = -\sqrt{3}b$ cm^2/s

Therefore, the area is decreasing at the rate of $\sqrt{3}b$ cm^2/s.

4. Find the equation of the normal to the curve $y^2 = 4x$ at the point (1, 2).

Ans. Equation of the curve is $y^2 = 4x$ ....(i)

$\Rightarrow 2y\frac{dy}{dx} = 4$

$\Rightarrow \frac{dy}{dx} = \frac{2}{y}$

$\therefore$ Slope of the tangent to the curve at the point (1, 2) to curve (i) is $\frac{2}{2}$ =1

$\therefore$ Slope of the normal to the curve at (1, 2) is $\frac{-1}{m} = \frac{-1}{1} = -1$

$\therefore$ Equation of the normal to the curve (i) at (1, 2) is $y-2=-1(x-1)$

$\Rightarrow y-2=-x+1$

$\Rightarrow x+y=3$

5. Show that the normal at any point θ to the curve $x=a\cos\theta+a\theta\sin\theta$, $y=a\sin\theta-a\theta\cos\theta$ is at a constant distance from the origin.

Ans. The parametric equations of the curve are

$x=a\cos\theta+a\theta\sin\theta$, $y=a\sin\theta+a\theta\cos\theta$

$\Rightarrow x=a(\cos\theta+\theta\sin\theta)$, $y=a(\sin\theta-\theta\cos\theta)$

$\therefore \dfrac{dx}{d\theta}=a[-\sin\theta+\theta\cos\theta+\sin\theta]=a\theta\cos\theta$

And $\dfrac{dy}{d\theta}=a[\cos\theta-(-\theta\sin\theta+\cos\theta)]=a[\cos\theta+\theta\sin\theta-\cos\theta]=a\theta\sin\theta$

$\therefore$ Slope of tangent at point (x,y)

$=\dfrac{dy}{dx}=\dfrac{dy/d\theta}{dx/d\theta}=\dfrac{a\theta\sin\theta}{a\theta\cos\theta}=\tan\theta$

$\therefore$ Slope of normal at any point θ

$=\dfrac{-1}{\tan\theta}=-\cot\theta=-\dfrac{\cos\theta}{\sin\theta}$

And Equation of normal at any point θ

i.e., at (x,y) = $[a(\cos\theta+\theta\sin\theta), a(\sin\theta-\theta\cos\theta)]$ is

$y-a(\sin\theta-\theta\cos\theta)=\dfrac{-\cos\theta}{\sin\theta}[x-a(\cos\theta+\theta\sin\theta)]$

$\Rightarrow y\sin\theta-a\sin^2\theta+a\theta\cos\theta\sin\theta=-x\cos\theta+a\cos^2\theta+a\theta\sin\theta\cos\theta$

$\Rightarrow x\cos\theta+y\sin\theta=a(\sin^2\theta+\cos^2\theta)$

$\Rightarrow x\cos\theta+y\sin\theta=a$

$\Rightarrow x\cos\theta+y\sin\theta-a=0$

$\therefore$ Distance of normal from origin (0, 0)

$=\dfrac{|0+0-a|}{\sqrt{\cos^2\theta+\sin^2\theta}}=a$ which is a constant.

6. Find the intervals in which the function f given by $f\left(x\right)=\dfrac{4\sin x-2x-x\cos x}{2+\cos x}$ is (i) increasing (ii) decreasing.

Ans. Given: $f\left(x\right)=\dfrac{4\sin x-2x-x\cos x}{2+\cos x}$

$=\dfrac{4\sin x-x(2+\cos x)}{2+\cos x}$

$=\dfrac{4\sin x}{2+\cos x}-\dfrac{x(2+\cos x)}{2+\cos x}$

$=\dfrac{4\sin x}{2+\cos x}-x$

$\therefore f'\left(x\right)=\dfrac{\left(2+\cos x\right)\dfrac{d}{dx}\left(4\sin x\right)-4\sin x\dfrac{d}{dx}\left(2+\cos x\right)}{\left(2+\cos x\right)^2}-1$

$\Rightarrow f'\left(x\right)=\dfrac{\left(2+\cos x\right)\left(4\cos x\right)-4\sin x\left(-\sin x\right)}{\left(2+\cos x\right)^2}-1$

$=\dfrac{8\cos x+4\cos^2 x+4\sin^2 x}{\left(2+\cos x\right)^2}-1$

$\Rightarrow f'\left(x\right)=\dfrac{8\cos x+4}{\left(2+\cos x\right)^2}-1$

$=\dfrac{8\cos x+4-\left(2+\cos x\right)^2}{\left(2+\cos x\right)^2}$

$=\dfrac{8\cos x+4--4-\cos^2 x-4\cos x}{\left(2+\cos x\right)^2}$

$\Rightarrow f'(x) = \frac{4\cos x - \cos^2 x}{(2+\cos x)^2}$

= $\cos x \frac{(4-\cos x)}{(2+\cos x)^2}$ ....(i)

Now $4-\cos x > 0$ for all real x as $-1 \le \cos x \le 1$. Also $(2+\cos x)^2 > 0$

(i) $f(x)$ is increasing if $f'(x) \ge 0$, i.e., from eq. (i), $\cos x \ge 0$

$\Rightarrow x$ lies in I and IV quadrants, i.e., $f(x)$ is increasing for $0 \le x \le \frac{\pi}{2}$

and $\frac{3\pi}{2} \le x \le 2\pi$

and (ii) $f(x)$ is decreasing if $f'(x) \le 0$, i.e., from eq. (i), $\cos x \le 0$

$\Rightarrow x$ lies in II and III quadrants, i.e., $f(x)$ is decreasing for $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

7. Find the intervals in which the function f given by $f(x) = x^3 + \frac{1}{x^3}, x \ne 0$ is (i) increasing (ii) decreasing.

Ans. (i) Given: $f(x) = x^3 + \frac{1}{x^3}, x \ne 0$

$\Rightarrow f(x) = x^3 + x^{-3}$

$\therefore f'(x) = 3x^2 - 3x^{-4} = 3\left(x^2 - \frac{1}{x^4}\right)$

= $3\left(\frac{x^6-1}{x^4}\right)$ = $\frac{3}{x^4}\left[(x^2)^3 - 1^3\right]$

$\Rightarrow f'(x) = \frac{3}{x^4}(x^2-1)(x^4+x^2+1)$

= $\frac{3}{x^4}(x^4+x^2+1)(x+1)(x-1)$ ....(i)

Now $f'(x)=0$

$\Rightarrow \frac{3}{x^4}(x^4+x^2+1)(x+1)(x-1)$ = 0

$\Rightarrow 3(x^4+x^2+1)(x+1)(x-1)$ = 0

Here, $3(x^4+x^2+1)$ is positive for all real $x \neq 0$

$\therefore x+1=0$ or $x-1=0$ [Turning points]

Therefore, $x=-1$ or $x=1$ divide the real line into three disjoint sub intervals $(-\infty,-1)$, $(-1,1)$ and $(1,\infty)$

For $(-\infty,-1)$, from eq. (i) at $x=-2$ (say),

$f'(x)=(+)(-)(-)=(+)$

Therefore, $f(x)$ is increasing at $(-\infty,-1)$

For $(-1,1)$, from eq. (i) at $x=\frac{1}{2}$ (say)

$f'(x)=(+)(+)(-)=(-)$

Therefore, $f(x)$ is decreasing at $(-1,1)$.

For $(1,\infty)$, from eq. (i) at $x=2$ (say),

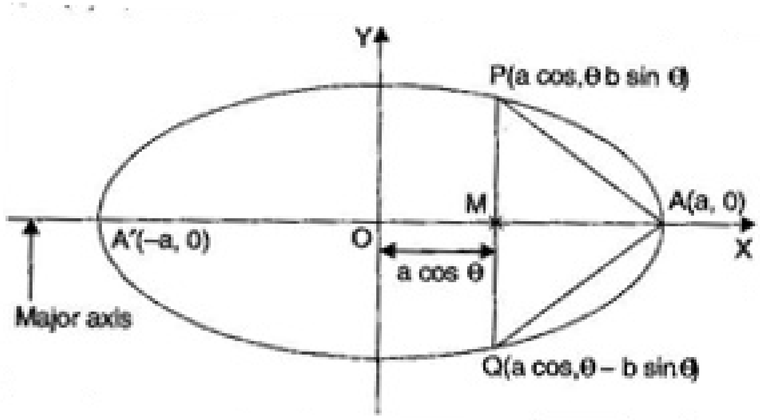
$f'(x)=(+)(+)(+)=(+)$

Therefore, $f(x)$ is increasing at $(1, \infty)$.

Therefore, $f(x)$ is (i) an increasing function for $x \leq -1$ and for $x \geq 1$ and (ii) decreasing function for $-1 \leq x \leq 1$.

8. Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with its vertex at one end of the major axis.

Ans. Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ....(i)



Comparing eq. (i) with $\cos^2\theta+\sin^2\theta=1$, we have $\frac{x}{a}=\cos\theta$ and $\frac{y}{b}=\sin\theta$

$\Rightarrow x=a\cos\theta$ and $y=b\sin\theta$

$\therefore$ Any point on ellipse is P$(a\cos\theta, b\sin\theta)$.

Draw PM perpendicular to $x-$ axis and produce it to meet the ellipse in the point Q.

$\therefore$ OM = $a\cos\theta$ and PM = $b\sin\theta$

We know that the ellipse (i) is symmetrical about $x-$ axis, therefore, PM = QM and hence triangle APQ is isosceles.

Area of ΔAPQ $(z)=\frac{1}{2}$ x Base x Height

= $\frac{1}{2}$ PQ.AM = $\frac{1}{2}$. 2PM.AM = PM (OA – OM)

$\Rightarrow z = b\sin\theta\left(a - a\cos\theta\right)$

= $ab\left(\sin\theta - \sin\theta\cos\theta\right)$

$\Rightarrow z = \frac{ab}{2}\left(2\sin\theta - 2\sin\theta\cos\theta\right)$

= $\frac{ab}{2}\left(2\sin\theta - \sin 2\theta\right)$

$\Rightarrow \frac{dz}{d\theta} = \frac{ab}{2}\left(2\cos\theta - 2\cos 2\theta\right)$

= $ab\left(\cos\theta - \cos 2\theta\right)$

$\Rightarrow \frac{d^2z}{d\theta^2} = ab\left(-\sin\theta + 2\sin 2\theta\right)$

Now $\frac{dz}{d\theta} = 0$

$\Rightarrow ab\left(\cos\theta - \cos 2\theta\right)$ = 0

$\Rightarrow \cos\theta - \cos 2\theta = 0$

$\Rightarrow \cos\theta = \cos 2\theta$

$\Rightarrow \cos\theta = \cos\left(360^\circ - 2\theta\right)$

$\Rightarrow \theta = 2\theta$ or $\theta = 360^\circ - 2\theta$

i.e., $\theta = 0$ or $3\theta = 360^\circ$

$\Rightarrow \theta = 120^\circ$

$\theta = 0$ is impossible

$\therefore \theta = 120°$

At $\theta = 120°$, $\frac{d^2 z}{d\theta^2} = ab\left(-\sin 120° + 2\sin 240°\right)$

= $ab\left(\frac{\sqrt{3}}{2} - \frac{2\sqrt{3}}{2}\right) = ab\left(\frac{-3\sqrt{3}}{2}\right)$ [Negative]

$\therefore z$ is maximum at $\theta = 120°$

$\therefore$ From eq. (i), Maximum area

= $\frac{ab}{2}\left(2\sin 120° - \sin 240°\right)$

= $\frac{ab}{2}\left(\frac{2\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right)$

= $\frac{ab}{2}\left(\frac{3\sqrt{3}}{2}\right)$ = $\frac{3\sqrt{3}}{4}ab$

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9. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs 70 per sq. meter for the base and 45 per square meter for sides. What is the cost of least expensive tank?

Ans. Given: Depth of tank = 2 m

Let x m be the length and y m be the breadth of the base of the tank.

Volume of tank = 8 cubic meters

$\Rightarrow x.y.2 = 8$

$\Rightarrow xy = 4$

$\Rightarrow\ y = \frac{4}{x}$

Cost of building the base of the tank at the rate of ` 70 per sq. meter is $70xy$.

And cost of building the four walls of the tank at the rate of ` 45 per sq. meter is

$45(x.2 + x.2 + y.2 + y.2)$

= ` $(180x + 180y)$

Let z be the total cost of building the tank.

$z = 70xy + 180x + 180y$

$= 70x\,(4/x) + 180x + 180\,(4/x)$

$= 280 + 180x + \frac{720}{x}$

$\therefore\ \frac{dz}{dx} = 0 + 180 - \frac{720}{x^2}$ and $\frac{d^2z}{dx^2} = \frac{1440}{x^3}$

Now $\frac{dz}{dx} = 0$

$\Rightarrow\ 180 - \frac{720}{x^2} = 0$

$\Rightarrow \frac{720}{x^2} = 180$

$\Rightarrow\ 180x^2 = 720$

$\Rightarrow\ x^2 = 4$

$\Rightarrow\ x = 2$ [length cannot be negative]

At $x=2$ $\frac{d^2z}{dx^2}=\frac{1440}{8}=180$ [Positive]

$\therefore$ z is minimum at $x=2$.

$\therefore$ Minimum cost = $280+180\times2+\frac{720}{2}$

= 280 + 360 + 360 = Rs 1000

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10. The sum of the perimeter of a circle and square is k, where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

Ans. Let x be the radius of the circle and y be the side of square.

According to question, Perimeter of circle + Perimeter of square = k

$\Rightarrow 2\pi x+4y=k$

$\Rightarrow 4y=k-2\pi x$

$\Rightarrow y=\frac{k-2\pi x}{4}$ ....(i)

Let z be the sum of areas of circle and square.

$\therefore z=\pi x^2+y^2$

$\Rightarrow z=\pi x^2+\frac{(k-2\pi x)^2}{16}$ [From eq. (i)]

$\Rightarrow z=\frac{16\pi x^2+k^2+4\pi^2x^2-4k\pi x}{16}$

= $\frac{1}{16}\left[\left(16\pi+4\pi^2\right)x^2-4k\pi x+k^2\right]$

$\therefore \frac{dz}{dx} = \frac{1}{16}\left[\left(16\pi + 4\pi^2\right)2x - 4k\pi\right]$ and $\frac{d^2z}{dx^2} = \frac{1}{16}\left(16\pi + 4\pi^2\right)2$

Now $\frac{dz}{dx} = 0$

$\Rightarrow \frac{1}{16}\left[\left(16\pi + 4\pi^2\right)2x - 4k\pi\right] = 0$

$\Rightarrow \left(16\pi + 4\pi^2\right)2x - 4k\pi = 0$

$\Rightarrow 4\pi(4+\pi)2x = 4k\pi$

$\Rightarrow x = \frac{4k\pi}{4\pi(4+\pi)2} = \frac{k}{2(4+\pi)}$

At $x = \frac{k}{2(4+\pi)}$ $\frac{d^2z}{dx^2} = \frac{1}{16}\left(16\pi + 4\pi^2\right)2$ [Positive]

$\therefore z$ is minimum when $x = \frac{k}{2(4+\pi)}$

$\therefore$ From eq. (i),

$y = \frac{1}{4}\left[k - 2\pi\frac{k}{2(4+\pi)}\right]$

= $\frac{1}{4}\left[k - \frac{\pi k}{4+\pi}\right]$

= $\frac{1}{4}\left[\frac{k(4+\pi) - \pi k}{4+\pi}\right]$

$\Rightarrow y = \frac{4k + \pi k - \pi k}{4(4+\pi)}$

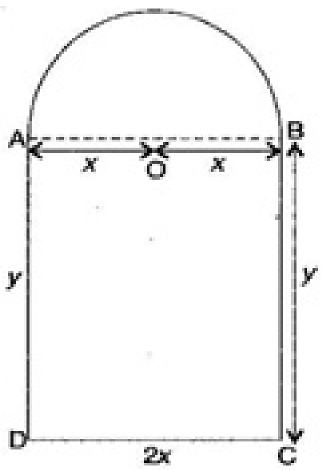
= $\dfrac{4k}{4(4+\pi)}$

= $2\dfrac{k}{2(4+\pi)}$ = $2x$

Therefore, sum of areas is minimum when side of the square is double the radius of the circle.

11. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

Ans. Let x m be the radius of the semi-circular opening of the window. Then one side of rectangle part of window is $2x$ and y m be the other side of rectangle.



$\therefore$ Perimeter of window

= Semi-circular arc AB + Length (AD + DC + BC)

$\Rightarrow \frac{1}{2}(2\pi x)+y+2x+y=10$

$\Rightarrow \pi x+2x+2y=10$

$\Rightarrow 2y=10-\pi x-2x$

$\Rightarrow\ y=\dfrac{10-(\pi+2)x}{2}$ ....(i)

Area of window (z)

= Area of semi-circle + Area of rectangle

= $\dfrac{1}{2}(\pi x^2)+2xy$

$\Rightarrow z=\dfrac{1}{2}\pi x^2+2x\left[\dfrac{10-(\pi+2)x}{2}\right]$

= $\dfrac{1}{2}\left[\pi x^2+20x-2(\pi+2)x^2\right]$

$\Rightarrow\ z=\dfrac{1}{2}\left[\pi x^2+20x-2\pi x^2-4x^2\right]$

= $\dfrac{1}{2}\left[-\pi x^2-4x^2+20x\right]$

$\therefore\ \dfrac{dz}{dx}=\dfrac{1}{2}[-2\pi x-8x+20]$ and $\dfrac{d^2z}{dx^2}=\dfrac{1}{2}(-2\pi-8)$ = $-(\pi+4)$

Now $\dfrac{dz}{dx}=0$

$\Rightarrow\ \dfrac{1}{2}[-2\pi x-8x+20]=0$

$\Rightarrow\ -2\pi x-8x+20=0$

$\Rightarrow\ -2x(\pi+4)=-20$

$\Rightarrow\ x=\dfrac{20}{2(\pi+4)}$

$\Rightarrow x=\frac{10}{\pi+4}$

At $x=\frac{10}{\pi+4}$ $\frac{d^2z}{dx^2}=-(\pi+4)$ [Negative]

$\therefore z$ is maximum at $x=\frac{10}{\pi+4}$

$\therefore$ From eq. (i), $y=\dfrac{10-(\pi+2)\frac{10}{\pi+4}}{2}$

$=\frac{10(\pi+4)-10(\pi+2)}{2(\pi+4)}$

$\Rightarrow y=\frac{10\pi+40-10\pi-20}{2(\pi+4)}$

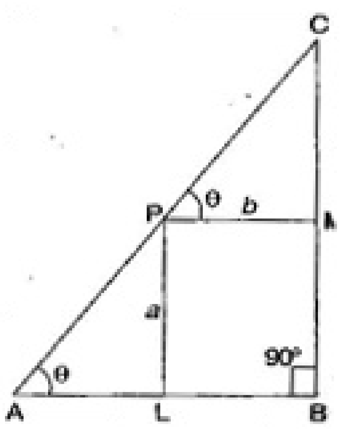
$=\frac{20}{2(\pi+4)}=\frac{10}{\pi+4}$ m

Therefore, Length of rectangle = $2x=\frac{20}{\pi+4}$ m and Width of rectangle = $y=\frac{10}{\pi+4}$ m

And Radius of semi-circle = $x=\frac{10}{\pi+4}$ m

12. A point on the hypotenuse of a triangle is at distances a and b from the sides of the triangle. Show that the maximum length of the hypotenuse is $\left(a^{2/3}+b^{2/3}\right)^{3/2}$.

Ans. Let P be a point on the hypotenuse AC of a right triangle ABC such that PL $\perp$ AB = a and PM $\perp$ BC = b and let $\angle$BAC = $\angle$MPC = θ, then in right angled ΔALP, $\frac{AP}{PL}=\operatorname{cosec}\theta$



$\therefore$ AP = PL $\operatorname{cosec}\theta$ = $a \operatorname{cosec}\theta$

And in right angled ΔPMC, $\frac{\text{PC}}{\text{PM}} = \sec\theta$

$\Rightarrow$ PM = PM $\sec\theta = b\sec\theta$

Let AC = z, then

z = AP + PC = $a\operatorname{cosec}\theta + b\sec\theta, 0 < \theta < \frac{\pi}{2}$ ....(i)

$\therefore \frac{dz}{d\theta} = -a\operatorname{cosec}\theta\cot\theta + b\sec\theta\tan\theta$

Now $\frac{dz}{d\theta} = 0$

$\Rightarrow -a\operatorname{cosec}\theta\cot\theta + b\sec\theta\tan\theta = 0$

$\Rightarrow \frac{b\sin\theta}{\cos^2\theta} = \frac{a\cos\theta}{\sin^2\theta}$

$\Rightarrow b\sin^3\theta = a\cos^3\theta \Rightarrow \frac{a}{b} = \frac{\sin^3\theta}{\cos^3\theta}$

$\Rightarrow \frac{a}{b} = \tan^3\theta$

$\Rightarrow \tan\theta = \left(\frac{a}{b}\right)^{\frac{1}{3}}$ ....(ii)

And

$$\frac{d^2z}{d\theta^2} = a\left[\operatorname{cosec}\theta\left(-\operatorname{cosec}^2\theta\right) + \cot\theta\left(-\operatorname{cosec}\theta\cot\theta\right)\right] + b\left[\sec\theta\sec^2\theta + \tan\theta\sec\theta\tan\theta\right]$$

$$\Rightarrow \frac{d^2z}{d\theta^2} = a\left(\operatorname{cosec}^3\theta + \operatorname{cosec}\theta\cot^2\theta\right) + b\left(\sec^3\theta + \sec\theta\tan^2\theta\right)$$

$\Rightarrow \frac{d^2z}{d\theta^2} > 0$ [$a > 0, b > 0$ and θ is +ve as $0 < \theta < \frac{\pi}{2}$)

$\therefore z$ is minimum when $\tan\theta = \left(\frac{a}{b}\right)^{\frac{1}{3}}$

$\therefore \sec^2\theta = 1 + \tan^2\theta = 1 + \left(\frac{a}{b}\right)^{\frac{2}{3}} = \frac{b^{\frac{2}{3}} + a^{\frac{2}{3}}}{b^{\frac{2}{3}}}$

$\Rightarrow \sec\theta = \frac{\left(b^{\frac{2}{3}} + a^{\frac{2}{3}}\right)^{\frac{1}{2}}}{b^{\frac{1}{3}}}$

Also $\operatorname{cosec}^2\theta = 1 + \cot^2\theta = 1 + \left(\frac{b}{a}\right)^{\frac{2}{3}}$

$= \frac{a^{\frac{2}{3}} + b^{\frac{2}{3}}}{a^{\frac{2}{3}}}$

$\Rightarrow$ $\operatorname{cosec}\theta = \frac{\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}}}{a^{\frac{1}{3}}}$

Putting these values in eq. (i),

Minimum length of hypotenuse = $a.\frac{\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}}}{a^{\frac{1}{3}}} + b.\frac{\left(b^{\frac{2}{3}} + a^{\frac{2}{3}}\right)^{\frac{1}{2}}}{b^{\frac{1}{3}}}$

= $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}}\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)$ = $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$

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13. Find the points at which the function f given by $f(x) = (x-2)^4(x+1)^3$ has:

(i) local maxima

(ii) local minima

(iii) point of inflexion.

Ans. Given: $f(x) = (x-2)^4(x+1)^3$ ....(i)

$\therefore f'(x) = (x-2)^4 \frac{d}{dx}(x+1)^3 + (x+1)^3 \frac{d}{dx}(x-2)^4$

= $3(x-2)^4(x+1)^2 + 4(x+1)^3(x-2)^3$

= $(x-2)^3(x+1)^2\left[3(x-2) + 4(x+1)\right]$

= $(x-2)^3(x+1)^2(7x-2)$

Now $f(x) = 0$

$\Rightarrow (x-2)^3 (x+1)^2 (7x-2)$ = 0

$\Rightarrow$ $x-2=0$ or $x+1=0$ or $7x-2=0$

$\Rightarrow$ $x=2$ or $x=-1$ or $x=\frac{2}{7}$

Now, for values of x close to $\frac{2}{7}$ and to the left of $\frac{2}{7}, f'(x)>0$. Also for values of x close to $\frac{2}{7}$ and to the right of $\frac{2}{7}, f'(x)<0$.

Therefore, $x=\frac{2}{7}$ is the point of local maxima.

Now, for values of x close to 2 and to the left of $2, f'(x)<0$. Also for values of x close to 2 and to the right of $2, f'(x)>0$.

Therefore, $x=2$ is the point of local minima.

Now as the values of x varies through $-1, f'(x)$ does not change its sign. Therefore, $x=-1$ is the point of inflexion.

14. Find the absolute maximum and minimum values of the function f given by $f(x)=\cos^2 x+\sin x, x\in[0,\pi]$.

Ans. Given: $f(x)=\cos^2 x+\sin x, x\in[0,\pi]$ ....(i)

$\therefore f'(x)=2\cos x\frac{d}{dx}\cos x+\cos x$

= $-2\cos x\sin x+\cos x$ = $\cos x(-2\sin x+1)$

Now $f'(x)=0$

$\Rightarrow \cos x(-2\sin x+1)$ = 0

$\Rightarrow \cos x = 0$ or $-2\sin x + 1 = 0$

$\Rightarrow x = \frac{\pi}{2}$ or $2\sin x = 1$

$\Rightarrow \sin x = \frac{1}{2}$

$\Rightarrow x = \frac{\pi}{6}$ [Turning points]

thus two turning points are $\pi/2 and \pi/6$

Now $f\left(\frac{\pi}{2}\right) = \cos^2\frac{\pi}{2} + \sin\frac{\pi}{2}$

= 0 + 1 = 1

$f\left(\frac{\pi}{6}\right) = \cos^2\frac{\pi}{6} + \sin\frac{\pi}{6} = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$

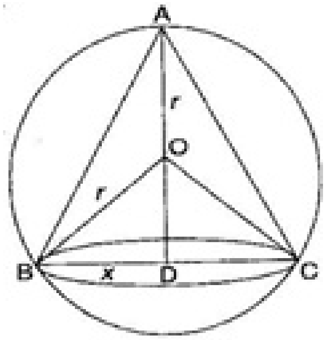
$f(0) = \cos^2 0 + \sin 0 = 1 + 0 = 1$

$f(\pi) = \cos^2\pi + \sin\pi = (-1)^2 + 0 = 1$

Therefore, absolute maximum is $\frac{5}{4}$ and absolute minimum is 1.

15. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

Ans. Let x be the radius of base of cone and y be the height of the cone inscribed in a sphere of radius r.



$\therefore$ OD = AD – AO = $y-r$

In right angled triangle OBD,

$OD^2 + BD^2 = OB^2$

$\Rightarrow (y-r)^2 + x^2 = r^2$

$\Rightarrow y^2 + r^2 - 2ry + x^2 = r^2$

$\Rightarrow x^2 = 2ry - y^2$ ....(i)

Volume of cone (V) = $\frac{1}{3}\pi x^2 y = \frac{1}{3}\pi\left(2ry - y^2\right)y$ [From eq. (i)]

$\Rightarrow$ V = $\frac{\pi}{3}\left(2ry^2 - y^3\right)$

$\Rightarrow \frac{dV}{dy} = \frac{\pi}{3}\left(4ry - 3y^2\right)$ and $\frac{d^2V}{dy^2} = \frac{\pi}{3}\left(4r - 6y\right)$

Now $\frac{dV}{dy} = 0$

$\Rightarrow \frac{\pi}{3}\left(4ry - 3y^2\right) = 0$

$\Rightarrow \frac{\pi y}{3}\left(4r - 3y\right) = 0$

$\Rightarrow 4r - 3y = 0$ and $y \neq 0$

$\Rightarrow\ y=\frac{4r}{3}$

At $y=\frac{4r}{3}\ \frac{d^2V}{dy^2}=\frac{\pi}{3}(4r-8r)$

= $\frac{-4\pi r}{3}$ [Negative]

$\therefore$ Volume is maximum at $y=\frac{4r}{3}$

16. Let f be a function defined on $[a,b]$ such that $f'(x)>0$, for all $x\in(a,b)$. Then prove that f is an increasing function on (a,b).

Ans. Let I be the interval (a,b)

Given: $f'(x)>0$ for all x in an interval I. Let $x_1,x_2\in$ I with $x_1<x_2$

By Lagrange's Mean Value Theorem, we have,

$\frac{f(x_2)-f(x_1)}{x_2-x_1}=f'(c)$, where $x_1<c<x_2$

$\Rightarrow f(x_2)-f(x_1)=(x_2-x_1)f'(c)$ where $x_1<c<x_2$

Now $x_1<x_2$

$\Rightarrow\ x_2-x_1>0$ ....(i)

Also, $f'(x)>0$ for all x in an interval I

$\Rightarrow\ f'(c)>0$

$\therefore$ From eq. (i), $f(x_2)-f(x_1)>0$

$\Rightarrow f(x_1) < f(x_2)$

Thus, for every pair of points $x_1, x_2 \in$ I with $x_1 < x_2$

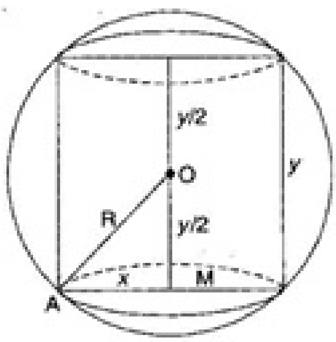
$\Rightarrow f(x_1) < f(x_2)$

Therefore, $f(x)$ is strictly increasing in I.

---

17. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

Ans. Let x be the radius and y be the height of the cylinder inscribed in a sphere having centre O and radius R. $(x > 0, y > 0)$



In right triangle OAM, $AM^2 + OM^2 = OA^2$

$\Rightarrow x^2 + \left(\frac{y}{2}\right)^2 = R^2$

$\Rightarrow x^2 = R^2 - \frac{y^2}{4}$ ....(i)

Volume of cylinder (V) = $\pi x^2 y$ ....(ii)

$\Rightarrow$ V = $\pi\left(R^2 - \frac{y^2}{4}\right)y$

= $\pi\left(R^2 y - \frac{y^3}{4}\right)$ ....(iii)

$\therefore \frac{dV}{dy} = \pi\left(R^2 - \frac{3y^2}{4}\right)$ and $\frac{d^2V}{dy^2} = \pi\left(-\frac{3y}{2}\right) = -\frac{3\pi y}{2}$

Now $\frac{dV}{dy} = 0$

$\Rightarrow \pi\left(R^2 - \frac{3y^2}{4}\right) = 0$

$\Rightarrow R^2 - \frac{3y^2}{4} = 0$

$\Rightarrow R^2 = \frac{3y^2}{4}$

$\Rightarrow y^2 = \frac{4R^2}{3}$

$\Rightarrow \ y = \frac{2R}{\sqrt{3}}$

At $y = \frac{2R}{\sqrt{3}}$ $\frac{d^2V}{dy^2} = -\frac{3\pi}{2}.\left(\frac{2R}{\sqrt{3}}\right)$

= $-\pi R\sqrt{3}$ [Negative]

$\therefore$ V is maximum at $y = \frac{2R}{\sqrt{3}}$

$\therefore$ From eq. (iii),

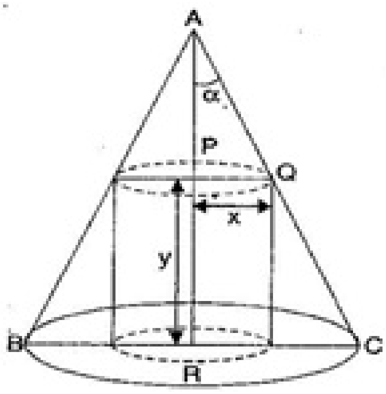
Maximum value of cylinder = $\pi\left[R^2.\frac{2R}{\sqrt{3}} - \frac{1}{4}.\frac{4R^2}{3}.\frac{2R}{\sqrt{3}}\right]$

= $\pi R^2 \frac{2R}{\sqrt{3}}\left[1-\frac{1}{3}\right]$

= $\frac{4\pi R^3}{3\sqrt{3}}$

18. Show that the height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and having semi-vertical angle α is one-third that of the cone and the greatest volume of the cylinder is $\frac{4}{27}\pi h^3 \tan^2 \alpha$.

Ans. Let r be the radius of the right circular cone of height h. Let the radius of the inscribed cylinder be x and height y.



In similar triangles APQ and ARC, $\frac{PQ}{RC}=\frac{AP}{AR}$

$\Rightarrow \frac{x}{r}=\frac{h-y}{h}$

$\Rightarrow hx = rh - ry$

$\Rightarrow ry = rh - hx = h(r-x)$

$\Rightarrow y = \frac{h}{r}(r-x)$

Volume of cylinder (V) = $\pi x^2 y$ ....(ii)

$\Rightarrow$ V = $\pi x^2 \frac{h}{r}(r-x)$

= $\frac{\pi h}{r}(rx^2 - x^3)$ ....(iii)

$\therefore \frac{dV}{dx} = \frac{\pi h}{r}(2rx - 3x^2)$ and $\frac{d^2V}{dx^2} = \frac{\pi h}{r}(2r - 6x)$

Now $\frac{dV}{dx} = 0$

$\Rightarrow \frac{\pi h}{r}(2rx - 3x^2) = 0$ and $x \neq 0$

$\Rightarrow 2rx - 3x^2 = 0$

$\Rightarrow 2r - 3x = 0$

$\Rightarrow x = \frac{2r}{3}$

At $x = \frac{2r}{3}$, $\frac{d^2V}{dx^2} = \frac{\pi h}{r}(2r - \frac{12r}{3})$

= $\frac{\pi h}{r}(-2r) = -2\pi h$ [Negative]

$\therefore$ V is maximum at $x = \frac{2r}{3}$

$\therefore$ From eq. (iii),

Maximum value of cylinder = $\frac{\pi h}{r}\left[r\frac{4r^2}{9} - \frac{8r^3}{27}\right]$

= $\frac{\pi h}{r} r^3 \left[\frac{4}{9} - \frac{8}{27}\right]$

= $\pi h r^2 . \frac{4}{27}$

$= \frac{4}{27}\pi h\left(h\tan\alpha\right)^2$

$= \frac{4}{27}\pi h^3\tan^2\alpha \left[\because \frac{r}{h} = \tan\alpha\right]$

Choose the correct answer in the Exercises 19 to 24:

19. A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic meter per hour. Then the depth of wheat is increasing at the rate of:

(A) 1 cu m/h

(B) 0.1 cu m/h

(C) 1.1 cu m/h

(D) 0.5 cu m/h

Ans. Let y be the depth of the wheat in the cylindrical tank of radius 10 m at time t.

$\therefore$ V = Volume of wheat in cylindrical tank at time $t = \pi\left(10\right)^2 y = 100\pi y$ cu. m

It is given that $\frac{dV}{dt}$ = 314 cu. m/hr

$\Rightarrow \frac{d}{dt}100\pi y = 314$

$\Rightarrow 100\pi\frac{dy}{dt} = 314$

$\Rightarrow 100(3.14)\frac{dy}{dt} = 314$

$\Rightarrow y = 1$ cu m/h

Therefore, option (A) is correct.

20. The slope of the tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point

$(2, -1)$ **is:**

(A) $\frac{22}{7}$

(B) $\frac{6}{7}$

(C) $\frac{7}{6}$

(D) $\frac{-6}{7}$

Ans. Equation of the curves are $x = t^2 + 3t - 8$ .....(i) and $y = 2t^2 - 2t - 5$ .....(ii)

$\therefore$ $\frac{dx}{dt} = 2t + 3$ and $\frac{dy}{dt} = 4t - 2$

$\therefore$ Slope of the tangent to the given curve at point (x, y) = $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t-2}{2t+3}$ .....(iii)

At the given point $(2, -1)$, $x = 2$ and $y = -1$

At $x = 2$, from eq. (i), $2 = t^2 + 3t - 8$

$\Rightarrow$ $t^2 + 3t - 10 = 0$

$\Rightarrow$ $(t+5)(t-2) = 0$

$\Rightarrow$ $t = -5, t = 2$

At $y = -1$, from eq. (ii), $-1 = 2t^2 - 2t - 5$

$\Rightarrow$ $2t^2 - 2t - 4 = 0$

$\Rightarrow$ $t^2 - t - 2 = 0$

$\Rightarrow$ $(t-2)(t+1) = 0$

$\Rightarrow\ t=2, t=-1$

Here, common value of t in the two sets of values is $t=2$

$\therefore$ From eq. (iii),

Slope of the tangent to the given curve at point $(2,-1)$ = $\frac{4(2)-2}{2(2)+3}=\frac{6}{7}$

Therefore, option (B) is correct.

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21. The line $y=mx+1$ is a tangent to the curve $y^2=4x$ if the value of m is:

(A) 1

(B) 2

(C) 3

(D) $\frac{1}{2}$

Ans. Equation of the curve is $y^2=4x$ ....(i)

$\therefore\ 2y\frac{dy}{dx}=4.1$

$\Rightarrow\ \frac{dy}{dx}=\frac{2}{y}$

$\therefore$ Slope of the tangent to the given curve at point (x,y) = $\frac{dy}{dx}=\frac{2}{y}$

$\therefore\ \frac{2}{y}=m$

$\Rightarrow\ y=\frac{2}{m}$ ....(ii)

Now $y = mx + 1$

$\Rightarrow \frac{2}{m} = mx + 1$

$\Rightarrow mx = \frac{2}{m} - 1$

$\Rightarrow x = \frac{2-m}{m}$ .....(iii)

Putting the values of x and y in eq. (i), $\frac{4}{m^2} = \frac{4(2-m)}{m^2}$

$\Rightarrow 2 - m = 1$

$\Rightarrow m = 1$

Therefore, option (A) is correct.

22. The normal at the point (1, 1) on the curve $2y + x^2 = 3$ is:

(A) $x + y = 0$

(B) $x - y = 0$

(C) $x + y + 1 = 0$

(D) $x - y = 1$

Ans. Equation of the given curve is $2y + x^2 = 3$ ....(i)

$\therefore \quad 2\frac{dy}{dx} + 2x = 0$

$\Rightarrow \frac{dy}{dx} = -x$

$\therefore$ Slope of the tangent to the given curve at point (1, 1) = $\frac{dy}{dx} = -x = -1 = m$ (say)

$\therefore$ Slope of the normal = $\frac{-1}{m} = \frac{-1}{-1} = 1$

$\therefore$ Equation of the normal at (1, 1) is $y - 1 = 1(x - 1)$

$\Rightarrow y - 1 = x - 1$

$\Rightarrow x - y = 0$

Therefore, option (B) is correct.

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23. The normal to the curve $x^2 = 4y$ passing through (1, 2) is:

(A) $x + y = 3$

(B) $x - y = 3$

(C) $x + y = 1$

(D) $x - y = 1$

Ans. Equation of the curve is $x^2 = 4y$ ....(i)

$\therefore 2x = 4\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{x}{2}$

$\therefore$ Slope of the normal at (x, y) = $-\frac{dx}{dy} = \frac{-2}{x}$ ....(ii)

Again slope of normal at given point (1, 2) = $\frac{y-2}{x-1}$ ....(iii)

From eq. (ii) and (iii), we have $\frac{-2}{x} = \frac{y-2}{x-1}$

$\Rightarrow -2x + 2 = xy - 2x$

$\Rightarrow xy = 2$

$\Rightarrow y = \frac{2}{x}$

$\therefore$ From eq. (i), $x^2 = \frac{8}{x}$

$\Rightarrow x^3 = 8$

$\Rightarrow x = 2$

$\therefore \; y = \frac{2}{x} = \frac{2}{2} = 1$

Now, at point (2, 1), slope of the normal from eq. (ii) = $\frac{-2}{x} = \frac{-2}{2} = -1$

$\therefore$ Equation of the normal is $y - 1 = -1(x - 2)$

$\Rightarrow y - 1 = -x + 2$

$\Rightarrow x + y = 3$

Therefore, option (A) is correct.

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24. The points on the curve $9y^2 = x^3$, where the normal to the curve make equal intercepts with axes are:

(A) $\left(4, \pm\frac{8}{3}\right)$

(B) $\left(4, -\frac{8}{3}\right)$

(C) $\left(4, \pm\frac{3}{8}\right)$

(D) $\left(\pm 4, \frac{3}{8}\right)$

Ans. Equation of the curve is $9y^2 = x^3$ ....(i)

$\therefore\ 18y\frac{dy}{dx} = 3x^2$

$\Rightarrow\ \frac{dy}{dx} = \frac{3x^2}{18y} = \frac{x^2}{6y}$

$\therefore$ Slope of the tangent to curve (i) at any point (x, y) = $\frac{dy}{dx} = \frac{x^2}{6y}$

$\therefore$ Slope of the normal = negative reciprocal = $\frac{-6y}{x^2} = \pm 1$

[$\because$ Slopes of lines making equal intercepts on the axes are ± 1]

$\Rightarrow\ -6y = \pm x^2$

Taking positive sign, $-6y = x^2$

$\Rightarrow\ y = \frac{-x^2}{6}$ ....(ii)

From eq. (i) and (ii),

$9y^2 = x^3$

$\therefore\quad 9\left(\frac{-x^2}{6}\right)^2 = x^3$

$\therefore$ x=4 and substituting this value of x in (i)

$\therefore \quad y = \frac{-8}{3}$

we have $x = 4$ and $y = \frac{-8}{3}$

Taking positive sign, $-6y = -x^2$

$\Rightarrow y = \frac{x^2}{6}$ ....(ii)

From eq. (i) and (ii),

$9y^2 = x^3$

$9\left(\frac{x^2}{6}\right)^2 = x^3$

$\Rightarrow$ x=4 and substituting in $y = \frac{x^2}{6}$

we have $x = 4$ and $y = \frac{8}{3}$

$\therefore$ Required points are $\left(4, \pm\frac{8}{3}\right)$

Therefore, option (A) is correct.