

CBSE Class-12 Mathematics

NCERT solution

Chapter - 5

Continuity & Differentiability - Exercise 5.7

Find the second order derivatives of the functions given in Exercises 1 to 5.

1. $x^2 + 3x + 2$

Ans. Let $y = x^2 + 3x + 2$

$$\therefore \frac{dy}{dx} = 2x + 3 \times 1 + 0 = 2x + 3$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = 2 \times 1 + 0 = 2$$

2. x^{20}

Ans. Let $y = x^{20}$

$$\therefore \frac{dy}{dx} = 20x^{19}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = 20 \times 19x^{18} = 380x^{18}$$

3. $x \cos x$

Ans. Let $y = x \cos x$

$$\therefore \frac{dy}{dx} = x \frac{d}{dx} \cos x + \cos x \frac{d}{dx} x = -x \sin x + \cos x$$

$$\begin{aligned}\Rightarrow \frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dx} \right) = -\frac{d}{dx} (x \sin x) + \frac{d}{dx} \cos x \\&= - \left[x \frac{d}{dx} \sin x + \sin x \frac{d}{dx} x \right] - \sin x \\&= - (x \cos x + \sin x) - \sin x \\&= -x \cos x - \sin x - \sin x \\&= -x \cos x - 2 \sin x \\&= - (x \cos x + 2 \sin x)\end{aligned}$$

4. $\log x$

Ans. Let $y = \log x$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{1}{x} \\ \Rightarrow \frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{1}{x} \right) = \frac{d}{dx} x^{-1} \\ \Rightarrow \frac{d^2y}{dx^2} &= (-1) x^{-2} = \frac{-1}{x^2}\end{aligned}$$

5. $x^3 \log x$

Ans. Let $y = x^3 \log x$

$$\begin{aligned}\therefore \frac{dy}{dx} &= x^3 \frac{d}{dx} \log x + \log x \frac{d}{dx} x^3 \\&= x^3 \cdot \frac{1}{x} + \log x (3x^2)\end{aligned}$$

$$= x^2 + 3x^2 \log x$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{d}{dx} (x^2 + 3x^2 \log x)$$

$$= \frac{d}{dx} x^2 + 3 \frac{d}{dx} (x^2 \log x)$$

$$= 2x + 3 \left[x^2 \frac{d}{dx} \log x + \log x \frac{d}{dx} x^2 \right]$$

$$= 2x + 3 \left(x^2 \cdot \frac{1}{x} + (\log x) 2x \right)$$

$$= 2x + 3(x + 2x \log x)$$

$$= 2x + 3x + 6x \log x$$

$$= 5x + 6x \log x$$

$$= x(5 + 6 \log x)$$

Find the second order derivatives of the functions given in Exercises 6 to 10.

6. $e^x \sin 5x$

Ans. Let $y = e^x \sin 5x$

$$\therefore \frac{dy}{dx} = e^x \frac{d}{dx} \sin 5x + \sin 5x \frac{d}{dx} e^x$$

$$= e^x \cos 5x \frac{d}{dx} 5x + \sin 5x \cdot e^x = e^x \cos 5x \times 5 + e^x \sin 5x$$

$$= e^x (5 \cos 5x + \sin 5x)$$

$$\begin{aligned}
 \Rightarrow \frac{d^2 y}{dx^2} &= e^x \frac{d}{dx} (5 \cos 5x + \sin 5x) + (5 \cos 5x + \sin 5x) \frac{d}{dx} e^x \\
 &= e^x [5(-\sin 5x) \times 5 + (\cos 5x) \times 5] + (5 \cos 5x + \sin 5x) e^x \\
 &= e^x (-25 \sin 5x + 5 \cos 5x + 5 \cos 5x + \sin 5x) \\
 &= e^x (10 \cos 5x - 24 \sin 5x) \\
 &= 2e^x (5 \cos 5x - 12 \sin 5x)
 \end{aligned}$$

7. $e^{6x} \cos 3x$

Ans. Let $y = e^{6x} \cos 3x$

$$\begin{aligned}
 \therefore \frac{dy}{dx} &= e^{6x} \frac{d}{dx} \cos 3x + \cos 3x \frac{d}{dx} e^{6x} \\
 &= e^{6x} (-\sin 3x) \frac{d}{dx} (3x) + \cos 3x \cdot e^{6x} \frac{d}{dx} (6x) \\
 &= -e^{6x} \sin 3x \times 3 + \cos 3x \cdot e^{6x} \times 6 \\
 &= e^{6x} (-3 \sin 3x + 6 \cos 3x) \\
 \Rightarrow \frac{d^2 y}{dx^2} &= e^{6x} \frac{d}{dx} (-3 \sin 3x + 6 \cos 3x) + (-3 \sin 3x + 6 \cos 3x) \frac{d}{dx} e^{6x} \\
 &= e^{6x} (-3 \cos 3x \times 3 - 6 \sin 3x \times 3) + (-3 \sin 3x + 6 \cos 3x) e^{6x} \times 6 \\
 &= e^{6x} (-9 \cos 3x - 18 \sin 3x - 18 \sin 3x + 36 \cos 3x) \\
 &= 9e^{6x} (3 \cos 3x - 4 \sin 3x)
 \end{aligned}$$

8. $\tan^{-1} x$

Ans. Let $y = \tan^{-1} x$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{1}{1+x^2} \\ \Rightarrow \frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{1}{1+x^2} \right) \\ &= \frac{(1+x^2) \frac{d}{dx}(1) - 1 \frac{d}{dx}(1+x^2)}{(1+x^2)^2} \\ &= \frac{(1+x^2) \times 0 - 2x}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2}\end{aligned}$$

9. $\log(\log x)$

Ans. Let $y = \log(\log x)$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{1}{\log x} \frac{d}{dx} \log x \left[\because \frac{d}{dx} \log f(x) = \frac{1}{f(x)} \frac{d}{dx} f(x) \right] \\ &= \frac{1}{\log x} \cdot \frac{1}{x} = \frac{1}{x \log x} \\ \Rightarrow \frac{d^2y}{dx^2} &= \frac{(x \log x) \frac{d}{dx}(1) - 1 \frac{d}{dx}(x \log x)}{(x \log x)^2} \\ &= \frac{(x \log x)(0) - \left[x \frac{d}{dx} \log x + \log x \frac{d}{dx} x \right]}{(x \log x)^2} \\ &= \frac{- \left[x \cdot \frac{1}{x} + \log x \times 1 \right]}{(x \log x)^2}\end{aligned}$$

$$= \frac{-[1+\log x]}{(x \log x)^2}$$

10. $\sin(\log x)$

Ans. Let $y = \sin(\log x)$

$$\therefore \frac{dy}{dx} = \cos(\log x) \frac{d}{dx}(\log x)$$

$$= \cos(\log x) \cdot \frac{1}{x} = \frac{\cos(\log x)}{x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{x \frac{d}{dx} \cos(\log x) - \cos(\log x) \frac{d}{dx} x}{x^2}$$

$$= \frac{x[-\sin(\log x)] \frac{d}{dx}(\log x) - \cos(\log x) \times 1}{x^2}$$

$$= \frac{-x \sin(\log x) \frac{1}{x} - \cos(\log x)}{x^2}$$

$$= \frac{-[\sin(\log x) + \cos(\log x)]}{x^2}$$

11. If $y = 5 \cos x - 3 \sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$.

Ans. Let $y = 5 \cos x - 3 \sin x$ (i)

$$\therefore \frac{dy}{dx} = -5 \sin x - 3 \cos x$$

$$\Rightarrow \frac{d^2 y}{dx^2} = -5 \cos x + 3 \sin x$$

$$= -(5 \cos x - 3 \sin x) = -y \text{ [From eq. (i)]}$$

$$\Rightarrow \frac{d^2 y}{dx^2} = -y$$

$$\Rightarrow \frac{d^2 y}{dx^2} + y = 0$$

12. If $y = \cos^{-1} x$. Find $\frac{d^2 y}{dx^2}$ in terms of y alone.

Ans. Given: $y = \cos^{-1} x$

$$\Rightarrow x = \cos y \quad \dots\dots\dots(i)$$

$$\therefore \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$= \frac{-1}{\sqrt{1-\cos^2 y}} \text{ [From eq. (i)]}$$

$$= \frac{-1}{\sqrt{\sin^2 y}} = \frac{-1}{\sin y} = -\operatorname{cosec} y \quad \dots\dots\dots(ii)$$

$$\Rightarrow \frac{d^2 y}{dx^2} = -\frac{d}{dx}(\operatorname{cosec} y)$$

$$= -\left[-\operatorname{cosec} y \cot y \frac{dy}{dx} \right]$$

$$= \operatorname{cosec} y \cot y (-\operatorname{cosec} y)$$

$$= -\operatorname{cosec}^2 y \cot y$$

[From eq. (ii)]

13. If $y = 3 \cos(\log x) + 4 \sin(\log x)$, **show that** $x^2 y_2 + x y_1 + y = 0$.

Ans. Given: $y = 3 \cos(\log x) + 4 \sin(\log x)$ (i)

$$\therefore \frac{dy}{dx} = y_1 = -3 \sin(\log x) \frac{d}{dx} \log x + 4 \cos(\log x) \frac{d}{dx} \log x$$

$$\Rightarrow y_1 = -3 \sin(\log x) \frac{1}{x} + 4 \cos(\log x) \frac{1}{x} = \frac{1}{x} [-3 \sin(\log x) + 4 \cos(\log x)]$$

$$\Rightarrow x y_1 = -3 \sin(\log x) + 4 \cos(\log x)$$

$$\text{Now } \frac{d}{dx}(x y_1) = -3 \cos(\log x) \frac{d}{dx} \log x - 4 \sin(\log x) \frac{d}{dx} \log x$$

$$\Rightarrow x \frac{d}{dx}(y_1) + y_1 \frac{d}{dx} x = -3 \cos(\log x) \frac{1}{x} - 4 \sin(\log x) \frac{1}{x}$$

$$\Rightarrow x y_2 + y_1 = - \frac{[3 \cos(\log x) + 4 \sin(\log x)]}{x}$$

$$\Rightarrow x(x y_2 + y_1) = -[3 \cos(\log x) + 4 \sin(\log x)]$$

$$\Rightarrow x(x y_2 + y_1) = -y \quad [\text{From eq. (i)}]$$

$$\Rightarrow x^2 y_2 + x y_1 + y = 0 \quad \text{Hence proved.}$$

14. If $y = A e^{mx} + B e^{nx}$, **show that** $\frac{d^2 y}{dx^2} - (m+n) \frac{dy}{dx} + mny = 0$.

Ans. Given: $y = A e^{mx} + B e^{nx}$ (i)

$$\text{To prove: } \frac{d^2 y}{dx^2} - (m+n) \frac{dy}{dx} + mny = 0$$

$$\therefore \frac{dy}{dx} = Ae^{mx} \frac{d}{dx}(mx) + Be^{nx} \frac{d}{dx}(nx) \left[\because \frac{d}{dx} e^{f(x)} = e^{f(x)} \frac{d}{dx} f(x) \right]$$

$$\Rightarrow \frac{dy}{dx} = Ame^{mx} + Bne^{nx} \dots(ii)$$

$$\text{Again } \frac{d^2y}{dx^2} = Ame^{mx} \cdot m + Bne^{nx} \cdot n$$

$$= Am^2e^{mx} + Bn^2e^{nx} \dots(iii)$$

$$\text{Now, L.H.S.} = \frac{d^2y}{dx^2} - (m+n) \frac{dy}{dx} + mny$$

$$= Am^2e^{mx} + Bn^2e^{nx} - (m+n)(Ame^{mx} + Bne^{nx}) + mn(Ae^{mx} + Be^{nx})$$

=

$$Am^2e^{mx} + Bn^2e^{nx} - Am^2e^{mx} - Bmne^{nx} - Amne^{mx} - Bn^2e^{nx} + Amne^{mx} + Bmne^{nx}$$

= 0

= R.H.S. Hence proved.

15. If $y = 500e^{7x} + 600e^{-7x}$ show that $\frac{d^2y}{dx^2} = 49y$.

Ans. Given: $y = 500e^{7x} + 600e^{-7x} \dots\dots(i)$

$$\therefore \frac{dy}{dx} = 500e^{7x}(7) + 600e^{-7x}(-7) = 500(7)e^{7x} - 600(7)e^{-7x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 500(7)e^{7x}(7) - 600(7)e^{-7x}(-7) = 500(49)e^{7x} + 600(49)e^{-7x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 49[500e^{7x} + 600e^{-7x}]$$

$$= 49y \text{ [From eq. (i)]}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 49y \text{ Hence proved.}$$

16. If $e^y(x+1) = 1$, show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

Ans. Given: $e^y(x+1) = 1$

$$\Rightarrow e^y = \frac{1}{x+1}$$

Taking log on both sides, $\log e^y = \log \frac{1}{x+1}$

$$\Rightarrow y \log e = \log 1 - \log(x+1)$$

$$\Rightarrow y = -\log(x+1)$$

$$\therefore \frac{dy}{dx} = -\frac{1}{x+1} \frac{d}{dx}(x+1) = \frac{-1}{x+1} = (x+1)^{-1}$$

$$\text{And } \frac{d^2y}{dx^2} = -(-1)(x+1)^{-2} \frac{d}{dx}(x+1) \left[\because \frac{d}{dx} \{f(x)\}^n = n \{f(x)\}^{n-1} \frac{d}{dx} f(x) \right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{(x+1)^2}$$

$$\text{Now, L.H.S.} = \frac{d^2y}{dx^2} = \frac{1}{(x+1)^2}$$

$$\text{And R.H.S.} = \left(\frac{dy}{dx}\right)^2 = \left(\frac{-1}{x+1}\right)^2 = \frac{1}{(x+1)^2}$$

\therefore L.H.S. = R.H.S. Hence proved.

17. If $y = (\tan^{-1} x)^2$, show that $(x^2 + 1)^2 y_2 + 2x(x^2 + 1) y_1 = 2$.

Ans. Given: $y = (\tan^{-1} x)^2$ (i)

$$\therefore y_1 = 2(\tan^{-1} x) \frac{d}{dx} \tan^{-1} x \left[\because \frac{d}{dx} \{f(x)\}^n = n\{f(x)\}^{n-1} \frac{d}{dx} f(x) \right]$$

$$\text{And } y_1 = 2(\tan^{-1} x) \frac{1}{1+x^2}$$

$$= \frac{2 \tan^{-1} x}{1+x^2}$$

$$\Rightarrow (1+x^2) y_1 = 2 \tan^{-1} x$$

Again differentiating both sides w.r.t. x ,

$$(1+x^2) \frac{d}{dx} y_1 + y_1 \frac{d}{dx} (1+x^2) = 2 \cdot \frac{1}{1+x^2}$$

$$\Rightarrow (1+x^2) y_2 + y_1 \cdot 2x = \frac{2}{1+x^2}$$

$$\Rightarrow (1+x^2)^2 y_2 + 2xy_1(1+x^2) = 2 \quad \text{Hence proved.}$$