CBSE Class–12 Mathematics

NCERT solution

Chapter - 5

Continuity & Differentiability - Exercise 5.1

1. Prove that the function $f(x)=5x-3$ is continuous at $x=0$, at $x=-3$ and at $x=5$.

Ans. Given: $f(x)=5x-3$

Continuity at $x=0$, $\lim_{x\to 0} f(x)=\lim_{x\to 0}(5x-3)$ = 5 (0) – 3 = 0 – 3 = – 3

And $f(0)=$ 5 (0) – 3 = 0 – 3 = – 3

Since $\lim_{x\to 0} f(x)=f(x)$, therefore, $f(x)$ is continuous at $x=0$.

Continuity at $x=-3$, $\lim_{x\to -3} f(x)=\lim_{x\to -3}(5x-3)$ = 5 (– 3) – 3 = – 15 – 3 = – 18

And $f(-3)=$ 5 (– 3) – 3 = – 15 – 3 = – 18

Since $\lim_{x\to -3} f(x)=f(x)$, therefore, $f(x)$ is continuous at $x=-3$

Continuity at $x=5$, $\lim_{x\to 5} f(x)=\lim_{x\to 5}(5x-3)$ = 5 (5) – 3 = 25 – 3 = 22

And $f(5)=$ 5 (5) – 3 = 25 – 3 = 22

Since $\lim_{x\to 5} f(x)=f(x)$, therefore, $f(x)$ is continuous at $x=5$.

2. Examine the continuity of the function $f(x)=2x^2-1$ at $x=3$.

Ans. Given: $f(x)=2x^2-1$

Continuity at $x=3$, $\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3}(2x^2-1) = 2(3)^2 - 1 = 18 - 1 = 17$

And $f(3) = 2(3)^2 - 1 = 18 - 1 = 17$

Since $\lim\limits_{x\to 3} f(x) = f(x)$, therefore, $f(x)$ is continuous at $x=3$.

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3. Examine the following functions for continuity:

(a) $f(x) = x - 5$

(b) $f(x) = \dfrac{1}{x-5}, x \neq 5$

(c) $f(x) = \dfrac{x^2-25}{x+5}, x \neq -5$

(d) $f(x) = |x-5|$

Ans. (a) Given: $f(x) = x - 5$

It is evident that f is defined at every real number k and its value at k is $k-5$.

It is also observed that $\lim\limits_{x\to k} f(x) = \lim\limits_{x\to k}(x-5) = k-5 = f(k)$

Since $\lim\limits_{x\to k} f(x) = f(k)$, therefore, $f(x)$ is continuous at every real number and it is a continuous function.

(b) Given: $f(x) = \dfrac{1}{x-5}, x \neq 5$

For any real number $k \neq 5$, we get $\lim\limits_{x\to k} f(x) = \lim\limits_{x\to k} \dfrac{1}{x-5} = \dfrac{1}{k-5}$

And $f(k) = \dfrac{1}{k-5}$

Since $\lim_{x\to k} f(x) = f(k)$, therefore, $f(x)$ is continuous at every point of domain of f and it is a continuous function.

(c) Given: $f(x) = \dfrac{x^2-25}{x+5}, x \neq -5$

For any real number $k \neq -5$, we get

$$\lim_{x\to k} f(x) = \lim_{x\to k}\frac{x^2-25}{x+5} = \lim_{x\to k}\frac{(x+5)(x-5)}{x+5} = \lim_{x\to k}(x-5) = k-5$$

And $f(k) = \dfrac{(k+5)(k-5)}{k+5} = k-5$

Since $\lim_{x\to k} f(x) = f(k)$, therefore, $f(x)$ is continuous at every point of domain of f and it is a continuous function.

(d) Given: $f(x) = |x-5|$

Domain of $f(x)$ is real and infinite for all real x

Here $f(x) = |x-5|$ is a modulus function.

Since, every modulus function is continuous, therefore, f is continuous in its domain R.

4. Prove that the function $f(x) = x^n$ is continuous at $x = n$ where n is a positive integer.

Ans. Given: $f(x) = x^n$ where n is a positive integer.

Continuity at $x = n$, $\lim_{x\to n} f(x) = \lim_{x\to n}(x^n) = n^n$

And $f(n) = n^n$

Since $\lim_{x\to n} f(x) = f(x)$, therefore, $f(x)$ is continuous at $x = n$.

5. Is the function f defined by $f(x)=\begin{cases} x, & \text{if } x\le 1 \\ 5, & \text{if } x>1 \end{cases}$ continuous at $x=0$, at $x=1$, at $x=2$?

Ans. Given: $f(x)=\begin{cases} x, & \text{if} \quad x\le 1 \\ 5, & \text{if} \quad x>1 \end{cases}$

At $x=0$, It is evident that f is defined at 0 and its value at 0 is 0.

Then $\lim_{x\to 0} f(x)=\lim_{x\to 0} x=0$ and $f(0)=0$

Therefore, $f(x)$ is continuous at $x=0$.

At $x=1$, Left Hand limit of f $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(x)=1$

Right Hand limit of f $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(x)=5$

Here $\lim_{x\to 1^-} f(x)\ne \lim_{x\to 1^+} f(x)$

Therefore, $f(x)$ is not continuous at $x=1$.

At $x=2$, f is defined at 2 and its value at 2 is 5.

$\lim_{x\to 2} f(x)=\lim_{x\to 2}(5)=5$, therefore, $\lim_{x\to 2} f(x)=f(2)$

Therefore, $f(x)$ is continuous at $x=2$.

Find all points of discontinuity of f, where f is defined by: (Exercise 6 to 12)

6. $f(x)=\begin{cases} 2x+3, & x\le 2 \\ 2x-3, & x>2 \end{cases}$

Ans. Given: $f(x)=\begin{cases}2x+3, & \text{if} \quad x\le 2\\ 2x-3, & \text{if} \quad x>2\end{cases}$

Here $f(x)$ is defined for $x\le 2$ i.e., on $(-\infty, 2)$ and also for $x>2$ i.e., on $(2,\infty)$.

$\therefore$ Domain of f is $(-\infty,2)\cup(2,\infty)=(-\infty,\infty)$ = R

$\therefore$ For all $x<2, f(x)=2x+3$ is a polynomial and hence continuous and for all $x>2, f(x)=2x-3$ is a continuous and hence it is also continuous on R – {2}.

Now Left Hand limit = $\lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^-}(2x+3)$ = 2 x 2 + 3 = 4 + 3 = 7

Right Hand limit = $\lim\limits_{x\to 2+} f(x)=\lim\limits_{x\to 2^+}(2x-3)$ = 2 x 2 – 3 = 4 – 3 = 1

Since $\lim\limits_{x\to 2^-} f(x)\ne\lim\limits_{x\to 2^+} f(x)$

Therefore, $\lim\limits_{x\to 2} f(x)$ does not exist and hence $f(x)$ is discontinuous at $x=2$ (only)

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7. $f(x)=\begin{cases}|x|+3, & \text{if } x\le -3\\ -2x, & \text{if } -3<x<3\\ 6x+2, & \text{if } x\ge 3\end{cases}$

Ans. Given: $f(x)=\begin{cases}|x|+3, & \text{if } x\le -3\\ -2x, & \text{if } -3<x<3\\ 6x+2, & \text{if } x\ge 3\end{cases}$

Here $f(x)$ is defined for $x\le -3$ i.e., on $(-\infty,-3)$ and for $-3<x<3$ i.e. on $(-3,3)$ and also for $x\ge 3$ i.e., on $(3,\infty)$.

$\therefore$ Domain of f is $(-\infty,-3)\cup(-3,3)\cup(3,\infty)=(-\infty,\infty)$ = R

$\therefore$ For all $x<-3, f(x)=|x|+3=-x+3$ is a polynomial and hence continuous and for all

$x(-3<x<3)$, $f(x)=-2x$ is a continuous and hence it is also continuous and also for all $x>3$, $f(x)=6x+2$. Therefore, $f(x)$ is continuous on R – $\{-3,3\}$.

It is observed that $x=-3$ and $x=3$ are partitioning points of domain R.

Now Left Hand limit = $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} (|x|+3) = \lim\limits_{x\to 3^-} (-x+3) = 3+3 = 6$

Right Hand limit = $\lim\limits_{x\to 3+} f(x) = \lim\limits_{x\to 3^+} (-2x) = (-2)(-3) = 6$

And $f(-3) = |-3|+3 = 3+3 = 6$

Therefore, $f(x)$ is continuous at $x=-3$.

Again Left Hand limit = $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} (-2x) = -2(3) = -6$

Right Hand limit = $\lim\limits_{x\to 3+} f(x) = \lim\limits_{x\to 3^+} (6x+2) = 6(3)+2 = 18+2 = 20$

Since $\lim\limits_{x\to 3^-} f(x) \neq \lim\limits_{x\to 3^+} f(x)$

Therefore, $\lim\limits_{x\to 3} f(x)$ does not exist and hence $f(x)$ is discontinuous at $x=3$ (only).

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8. $f(x)=\begin{cases} \dfrac{|x|}{x}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$

Ans. Given: $f(x)=\begin{cases} \dfrac{|x|}{x}, & \text{if } \quad x\neq 0 \\ 0, & \text{if } \quad x=0 \end{cases}$ i.e., $\dfrac{x}{x}=1$ if $x>0$ and $\dfrac{-x}{x}=-1$ if $x<0$

$\Rightarrow$ $f(x)=1$ if $x>0$, $f(x)=-1$ if $x<0$ and $f(x)=0$ if $x=0$

It is clear that domain of $f(x)$ is R as $f(x)$ is defined for $x>0$, $x<0$ and $x=0$.

For all $x > 0$, $f(x) = 1$ is a constant function and hence continuous.

For all $x < 0$, $f(x) = -1$ is a constant function and hence continuous.

Therefore $f(x)$ is continuous on R – {0}.

Now Left Hand limit = $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-1) = -1$

Right Hand limit = $\lim_{x \to 0+} f(x) = \lim_{x \to 0^+} (1) = 1$

Since $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

Therefore, $\lim_{x \to 0} f(x)$ does not exist and hence $f(x)$ is discontinuous at $x = 0$ (only).

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9. $f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$

Ans. Given: $f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{if} \quad x < 0 \\ -1, & \text{if} \quad x \geq 0 \end{cases}$

At $x = 0$, L.H.L. = $\lim_{x \to 0^-} \dfrac{x}{|x|} = -1$ And $f(0) = -1$

R.H.L. = $\lim_{x \to 0^+} f(x) = -1$

Since L.H.L. = R.H.L. = $f(0)$

Therefore, $f(x)$ is a continuous function.

Now, for $x = c < 0$ $\lim_{x \to c^-} \dfrac{x}{|x|} = -1 = f(c)$

$\therefore \lim\limits_{x \to c^-} = f(x)$

Therefore, $f(x)$ is a continuous at $x = c < 0$

Now, for $x = c > 0$ $\lim\limits_{x \to c^-} f(x) = 1 = f(c)$

Therefore, $f(x)$ is a continuous at $x = c > 0$

Hence, the function is continuous at all points of its domain.

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10. $f(x) = \begin{cases} x+1, & \text{if } x \geq 1 \\ x^2+1, & \text{if } x < 1 \end{cases}$

Ans. Given: $f(x) = \begin{cases} x+1, & \text{if} \quad x \geq 1 \\ x^2+1, & \text{if} \quad x < 1 \end{cases}$

It is observed that $f(x)$ being polynomial is continuous for $x \geq 1$ and $x < 1$ for all $x \in$ R.

Continuity at $x = 1$, R.H.L. = $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (x+1) = \lim\limits_{h \to 0} (1+h+1) = 2$

L.H.L. = $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (x^2+1) = \lim\limits_{h \to 0} \left((1-h)^2+1\right) = 2$

And $f(1) = 2$

Since L.H.L. = R.H.L. = $f(1)$

Therefore, $f(x)$ is a continuous at $x = 1$ for all $x \in$ R.

Hence, $f(x)$ has no point of discontinuity.

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11. $f(x) = \begin{cases} x^3-3, & \text{if } x \leq 2 \\ x^2+1, & \text{if } x > 2 \end{cases}$

Ans. Given: $f(x)=\begin{cases} x^3-3, & \text{if} \quad x\le 2 \\ x^2+1, & \text{if} \quad x>2 \end{cases}$

At $x=2$, L.H.L. = $\lim\limits_{x\to 2^-}(x^3-3)=8-3=5$

R.H.L. = $\lim\limits_{x\to 2^+}(x^2+1)=4+1=5$

$f(2)=2^3-3=8-3=5$

Since L.H.L. = R.H.L. = $f(2)$

Therefore, $f(x)$ is a continuous at $x=2$

Now, for $x=c<0$ $\lim\limits_{x\to c}(x^3-3)=c^3-3=f(c)$ and $\lim\limits_{x\to c}(x^2+1)=c^2+1=f(c)$

$\therefore \lim\limits_{x\to c^-}=f(x)$

Therefore, $f(x)$ is a continuous for all $x\in$ R.

Hence the function has no point of discontinuity.

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12. $f(x)=\begin{cases} x^{10}-1, & \text{if } x\le 1 \\ x^2, & \text{if } x>1 \end{cases}$

Ans. Given: $f(x)=\begin{cases} x^{10}-1, & \text{if} \quad x\le 1 \\ x^2, & \text{if} \quad x>1 \end{cases}$

At $x=1$, L.H.L. = $\lim\limits_{x\to 1^-}f(x)=\lim\limits_{x\to 1^-}(x^{10}-1)=0$

R.H.L. = $\lim\limits_{x\to 1^+}f(x)=\lim\limits_{x\to 1^+}(x^2)=1$

$f(1)=1^{10}-1=0$

Since L.H.L. $\neq$ R.H.L.

Therefore, $f(x)$ is discontinuous at $x=1$

Now, for $x=c<1$ $\lim\limits_{x\to c}\left(x^{10}-1\right)=c^{10}-1=f(c)$ and for $x=c>1$ $\lim\limits_{x\to c}\left(x^2\right)=c^2=f(c)$

Therefore, $f(x)$ is a continuous for all $x \in$ R – {1}

Hence for all given function $x=1$ is a point of discontinuity.

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13. Is the function defined by $f(x)=\begin{cases} x+5, & \text{if } x\leq 1 \\ x-5, & \text{if } x>1 \end{cases}$ **a continuous function?**

Ans. Given: $f(x)=\begin{cases} x+5, & \text{if } x\leq 1 \\ x-5, & \text{if } x>1 \end{cases}$

At $x=1$, L.H.L. = $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}(x+5)=6$

R.H.L. = $\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}(x-5)=-4$

Since L.H.L. $\neq$ R.H.L.

Therefore, $f(x)$ is discontinuous at $x=1$

Now, for $x=c<1$ $\lim\limits_{x\to c}(x+5)=c+5=f(c)$ and for $x=c>1$ $\lim\limits_{x\to c}(x-5)=c-5=f(c)$

Therefore, $f(x)$ is a continuous for all $x \in$ R – {1}

Hence $f(x)$ is not a continuous function.

Discuss the continuity of the function f, where f is defined by:

14. $f(x)=\begin{cases}3, & \text{if } 0\le x\le 1\\ 4, & \text{if } 1<x<3\\ 5, & \text{if } 3\le x\le 10\end{cases}$

Ans. Given: $f(x)=\begin{cases}3, & \text{if } 0\le x\le 1\\ 4, & \text{if } 1<x<3\\ 5, & \text{if } 3\le x\le 10\end{cases}$

In the interval $0\le x\le 1$, $f(x)=3$

$\therefore$ f is continuous in this interval.

At $x=1$, L.H.L. = $\lim\limits_{x\to 1^-} f(x)=3$ and R.H.L. = $\lim\limits_{x\to 1^+} f(x)=4$

Since L.H.L. $\neq$ R.H.L.

Therefore, $f(x)$ is discontinuous at $x=1$

At $x=3$, L.H.L. = $\lim\limits_{x\to 3^-} f(x)=4$ and R.H.L. = $\lim\limits_{x\to 3^+} f(x)=5$

Since L.H.L. $\neq$ R.H.L.

Therefore, $f(x)$ is discontinuous at $x=3$

Hence, f is discontinuous at $x=1$ and $x=3$.

15. $f(x)=\begin{cases}2x, & \text{if } x<0\\ 0, & \text{if } 0\le x\le 1\\ 4x, & \text{if } x>1\end{cases}$

Ans. Given: $f(x)=\begin{cases}2x, & \text{if } x<0\\ 0, & \text{if } 0\le x\le 1\\ 4x, & \text{if } x>1\end{cases}$

At $x=0$, L.H.L. = $\lim_{x\to 0^-} 2x=0$ and R.H.L. = $\lim_{x\to 0^+} (0)=0$

Since L.H.L. = R.H.L.

Therefore, $f(x)$ is continuous at $x=0$

At $x=1$, L.H.L. = $\lim_{x\to 1^-} (0)=0$ and R.H.L. = $\lim_{x\to 1^+} (4x)=4$

Since L.H.L. $\neq$ R.H.L.

Therefore, $f(x)$ is discontinuous at $x=1$

When $x<0$, $f(x)$ being a polynomial function is continuous for all $x<0$.

When $x>1$, $f(x)=4x$. It is being a polynomial function is continuous for all $x>1$.

Hence $x=1$ is a point of discontinuity.

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16. $f(x)=\begin{cases} -2, & \text{if } x\le -1 \\ 2x, & \text{if } -1<x\le 1 \\ 2, & \text{if } x>1 \end{cases}$

Ans. Given: $f(x)=\begin{cases} -2, & \text{if } \quad x\le -1 \\ 2x, & \text{if } \quad -1<x\le 1 \\ 2, & \text{if } \quad x>1 \end{cases}$

At $x=-1$, L.H.L. = $\lim_{x\to -1^-} f(x)=-2$ and R.H.L. = $\lim_{x\to -10^+} f(x)=-2$

Since L.H.L. = R.H.L.

Therefore, $f(x)$ is continuous at $x=-1$

At $x=1$, L.H.L. = $\lim_{x\to 1^-} f(x)=2$ and R.H.L. = $\lim_{x\to 1^+} f(x)=2$

Since L.H.L. = R.H.L.

Therefore, $f(x)$ is continuous at $x=1$

17. Find the relationship between a and b so that the function f defined by $f(x)=\begin{cases} ax+1, & \text{if} \quad x\le 3 \\ bx+3, & \text{if} \quad x>3 \end{cases}$ is continuous at $x=3$.

Ans. Given: $f(x)=\begin{cases} ax+1, & \text{if} \quad x\le 3 \\ bx+3, & \text{if} \quad x>3 \end{cases}$

Continuity at $x=3$,

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}(ax+1)=\lim_{h\to 0}\{a(3-h)+1\}=\lim_{h\to 0}(3a-ah+1)=3a+1$$

$$\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+}(bx+3)=\lim_{h\to 0}\{b(3+h)+3\}=\lim_{h\to 0}(3b+bh+3)=3b+3$$

Also $f(3)=3a+1$

$$\therefore \lim_{x\to 3^-} f(x)=\lim_{x\to 3^+} f(x)=f(3)$$

$$\Rightarrow 3b+3=3a+1$$

$$\Rightarrow 2=3(a-b)$$

$$\Rightarrow a-b=\frac{2}{3}$$

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18. For what value of λ is the function defined by $f(x)=\begin{cases} \lambda(x^2-2x), & \text{if} \quad x\le 0 \\ 4x+1, & \text{if} \quad x>0 \end{cases}$ continuous at $x=0$? What about continuity at $x=1$?

Ans. Since $f(x)$ is continuous at $x=0$.

$$\therefore \lim_{x\to 0^-} f(x)=f(0)=\lambda(x^2-2x)=\lambda(0-0)=0$$

And $\lim\limits_{x \to 0^+} f(x) = f(0) = 4x + 1 = 4 \times 0 + 1 = 1$

Here, therefore should be L.H.L. = R.H.L.

$\Rightarrow$ 0 = 1, which is not possible.

$\therefore$ for no value of λ, f(x) is continuous at x = 0

Again Since $f(x)$ is continuous at $x = 1$.

$=> \lim\limits_{x \to 1^-} f(x) = f(-1)$

$=> \lim\limits_{x \to 1^-} 4x + 1$

$=> \lim\limits_{h \to 0} 4(1 - h) + 1$

$=> 4 \times 1 + 1$

=> 5

And $\lim\limits_{x \to 1^+} f(x) = f(1) = 4x + 1 = 4 \times 1 + 1 = 5$

Here, L.H.L. = R.H.L.

$\therefore$ for any value of λ, f(x) is continuous at x = 1.

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19. Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. Here $[x]$ denotes the greatest integer less than or equal to x.

Ans. For any real number x we use the symbol $[x]$ to denote the fractional part or decimal part of x. For example,

[3.45] = 0.45

[–7.25] = 0.25

[3] = 0

[–7] = 0

The function g : R $\rightarrow$ R defined by $g(x) = x - [x] \forall x \in \infty$ is called the fractional part function. It is observed that the domain of the fractional part function is the set R of all real numbers and the range of the set [0, 1).

Hence given function is discontinuous function.

20. Is the function $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$?

Ans. Given: $f(x) = x^2 - \sin x + 5$

L.H.L. = $\lim\limits_{x \to \pi^-} \left(x^2 - \sin x + 5\right)$ = $\lim\limits_{h \to 0} \left[(\pi - h)^2 - \sin(\pi - h) + 5\right] = \pi^2 + 5$

R.H.L. = $\lim\limits_{x \to \pi^+} \left(x^2 - \sin x + 5\right)$ = $\lim\limits_{h \to 0} \left[(\pi + h)^2 - \sin(\pi + h) + 5\right] = \pi^2 + 5$

And $f(\pi) = \pi^2 - \sin \pi + 5 = \pi^2 + 5$

Since L.H.L. = R.H.L. = $f(\pi)$

Therefore, f is continuous at $x = \pi$

21. Discuss the continuity of the following functions:

(a) $f(x) = \sin x + \cos x$

(b) $f(x) = \sin x - \cos x$

(c) $f(x) = \sin x . \cos x$

Ans. (a) Let a be an arbitrary real number then $\lim\limits_{x \to a^-} f(x) \Rightarrow \lim\limits_{h \to 0} f(a + h)$

$\Rightarrow \lim\limits_{h \to 0} \sin(a + h) + \cos(a + h)$

$\Rightarrow \lim\limits_{h \to 0} \left(\sin a \cos h + \cos a \sin h + \cos a \cos h - \sin a \sin h \right)$

= $\sin a \cos 0 + \cos a \sin 0 + \cos a \cos 0 - \sin a \sin 0$

= $\sin a + 0 + \cos a - 0$

= $\sin a + \cos a = f\left(a\right)$

Similarly, we have $\lim\limits_{x \to a^-} f\left(x\right) = f\left(a\right)$

$\therefore \lim\limits_{x \to a^-} f\left(x\right) = f\left(a\right) = \lim\limits_{x \to a^+} f\left(x\right)$

Therefore, $f\left(x\right)$ is continuous at $x = a$.

Since, a is an arbitrary real number, therefore, $f\left(x\right) = \sin x + \cos x$ is continuous.

(b) Let a be an arbitrary real number then $\lim\limits_{x \to a^-} f\left(x\right) \Rightarrow \lim\limits_{h \to 0} f\left(a + h\right)$

$\Rightarrow \lim\limits_{h \to 0} \sin\left(a + h\right) - \cos\left(a - h\right)$

$\Rightarrow \lim\limits_{h \to 0} \left(\sin a \cos h + \cos a \sin h - \cos a \cos h - \sin a \sin h \right)$

= $\sin a \cos 0 + \cos a \sin 0 - \cos a \cos 0 - \sin a \sin 0$

= $\sin a + 0 - \cos a - 0$

= $\sin a - \cos a = f\left(a\right)$

Similarly, we have $\lim\limits_{x \to a^-} f\left(x\right) = f\left(a\right)$

$\therefore \lim\limits_{x \to a^-} f\left(x\right) = f\left(a\right) = \lim\limits_{x \to a^+} f\left(x\right)$

Therefore, $f\left(x\right)$ is continuous at $x = a$.

Since, a is an arbitrary real number, therefore, $f\left(x\right) = \sin x - \cos x$ is continuous.

(c) Let a be an arbitrary real number then $\lim\limits_{x \to a^+} f(x) \Rightarrow \lim\limits_{h \to 0} f(a+h)$

$\Rightarrow \lim\limits_{h \to 0} \sin(a+h).\cos(a+h)$

$\Rightarrow \lim\limits_{h \to 0} (\sin a \cos\ h + \cos a \sin\ h)(\cos a \cos\ h - \sin a \sin\ h)$

= $(\sin a \cos 0 + \cos a \sin 0)(\cos a \cos 0 - \sin a \sin 0)$

= $(\sin a + 0)(\cos a - 0)$

= $\sin a.\cos a = f(a)$

Similarly, we have $\lim\limits_{x \to a^-} f(x) = f(a)$

$\therefore \lim\limits_{x \to a^-} f(x) = f(a) = \lim\limits_{x \to a^+} f(x)$

Therefore, $f(x)$ is continuous at $x = a$.

Since, a is an arbitrary real number, therefore, $f(x) = \sin x.\cos x$ is continuous.

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22. Discuss the continuity of cosine, cosecant, secant and cotangent functions.

Ans. (a) Let a be an arbitrary real number then $\lim\limits_{x \to a^-} f(x) \Rightarrow \lim\limits_{x \to a^-} \cos x \Rightarrow \lim\limits_{h \to 0} \cos(a+h)$

$\Rightarrow \lim\limits_{h \to 0} (\cos a \cos\ h - \sin a \sin\ h)$

= $\cos a \lim\limits_{h \to 0} \cos\ h - \sin a \lim\limits_{h \to 0} \sin\ h$

= $\cos a \times 1 - \sin a \times 0$ = $\cos a = f(a)$

$\therefore \lim\limits_{x \to a} f(x) = f(a)$ for all $a \in$ R

Therefore, $f(x)$ is continuous at $x = a$.

Since, a is an arbitrary real number, therefore, $\cos x$ is continuous.

(b) $f(x) = \cos ec\ x = \frac{1}{\sin x}$ and domain $x = \mathrm{R} - (x\pi), x \in \mathrm{I}$

$$\Rightarrow \lim_{x \to a} \frac{1}{\sin x} = \frac{1}{\lim_{h \to 0} \sin(a+h)}$$

$$= \frac{1}{\lim_{h \to 0}(\sin a \cos\ h + \cos a \sin\ h)}$$

$$= \frac{1}{\sin a \cos 0 + \cos a \sin 0}$$

$$= \frac{1}{\sin a(1) + \cos a(0)}$$

$$= \frac{1}{\sin a} = f(a)$$

Therefore, $f(x)$ is continuous at $x = a$.

Since, a is an arbitrary real number, therefore, $f(x) = \cos ec\ x$ is continuous.

(c) $f(x) = \sec\ x = \frac{1}{\cos x}$ and domain $x = \mathrm{R} - (2x+1)\frac{\pi}{2}, x \in \mathrm{I}$

$$\Rightarrow \lim_{x \to a} \frac{1}{\cos x} = \frac{1}{\lim_{h \to 0} \cos(a+h)}$$

$$= \frac{1}{\lim_{h \to 0}(\cos a \cos\ h - \sin a \sin\ h)}$$

= $\dfrac{1}{\cos a \cos 0 - \sin a \sin 0}$

= $\dfrac{1}{\cos a(1) - \sin a(0)}$ = $\dfrac{1}{\cos a} = f(a)$

Therefore, $f(x)$ is continuous at $x = a$.

Since, a is an arbitrary real number, therefore, $f(x) = \sec x$ is continuous.

(d) $f(x) = \cot x = \dfrac{1}{\tan x}$ and domain $x = \mathrm{R} - (x\pi), x \in \mathrm{I}$

$\Rightarrow \lim\limits_{x\to a} \dfrac{1}{\tan x} = \dfrac{1}{\lim\limits_{h\to 0} \tan(a+h)}$

= $\dfrac{1}{\lim\limits_{h\to 0}\left(\dfrac{\tan a + \tan h}{1 - \tan a \tan h}\right)}$ = $\dfrac{1}{\dfrac{\tan a + 0}{1 - \tan a \tan 0}}$

= $\dfrac{1-0}{\tan a}$ = $\dfrac{1}{\tan a} = f(a)$

Therefore, $f(x)$ is continuous at $x = a$.

Since, a is an arbitrary real number, therefore, $f(x) = \cot x$ is continuous.

23. Find all points of discontinuity of f, where $f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x < 0 \\ x+1, & \text{if } x \geq 0 \end{cases}$.

Ans. Given: $f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x < 0 \\ x+1, & \text{if } x \geq 0 \end{cases}$

At $x=0$, L.H.L. = $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sin(-h)}{-h} = 1$

R.H.L. = $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x+1) = 0+1 = 1$

$f(0) = 1$

$\therefore$ f is continuous at $x=0$.

When $x<0$, $\sin x$ and x are continuous, then $\frac{\sin x}{x}$ is also continuous.

When $x>0$, $f(x) = x+1$ is a polynomial, then f is continuous.

Therefore, f is continuous at any point.

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24. Determine if f defined by $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ is a continuous function.

Ans. Here, $\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin \frac{1}{x}$ = 0 x a finite quantity = 0

$\left[\because \sin \frac{1}{x} \text{ lies between } -1 \text{ and } 1 \right]$

Also $f(0) = 0$

Since, $\lim_{x \to 0} f(x) = f(0)$, therefore, the function f is continuous at $x=0$.

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25. Examine the continuity of f, where f is defined by $f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1, & \text{if } x = 0 \end{cases}$.

Ans. The given function f is f(x)

$$= \begin{cases} \sin x - \cos x, if\, x \neq 0 \\ -1 \qquad\qquad if\, x = 0 \end{cases}$$

It is evident that f is defined at all point of the real line.

Let c be a real number.

Case I:

If c$\neq$0, then f(c) = sin c – cos c

$$\lim_{x\to c} f(x) = \lim_{x\to c}(\sin x - \cos x) = \sin c - \cos\, c$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all point x, such that x$\neq$ 0

Case II:

If c = 0, then f(0)=-1

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}(\sin x - \cos x) = \sin 0 - \cos\, 0$$

$$= 0 - 1 = -1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0}(\sin x - \cos x) = \sin 0 - \cos\, 0$$

$$= 0 - 1 = -1$$

$$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

Therefore, f is continuous at x=0

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Find the values of k so that the function f is continuous at the indicated point in Exercise 26 to 29.

26. $f(x)=\begin{cases}\dfrac{k\cos x}{\pi-2x}, & \text{if } x\neq\dfrac{\pi}{2}\\ 3, & \text{if } x=\dfrac{\pi}{2}\end{cases}$ **at** $x=\dfrac{\pi}{2}$.

Ans. Here, $\lim\limits_{x\to\frac{\pi}{2}} f(x)=\lim\limits_{x\to\frac{\pi}{2}}\dfrac{k\cos x}{\pi-2x}$

$\because\ x\to\dfrac{\pi}{2}$

$\Rightarrow\ x\neq\dfrac{\pi}{2}$

Putting $x=\dfrac{\pi}{2}+h$ where $h\to 0$

$=\lim\limits_{h\to 0}\dfrac{k\cos\left(\dfrac{\pi}{2}+h\right)}{\pi-2\left(\dfrac{\pi}{2}+h\right)}=\lim\limits_{h\to 0}\dfrac{-k\sin h}{\pi-\pi-2h}$

$=\lim\limits_{h\to 0}\dfrac{-k\sin h}{-2h}=\dfrac{k}{2}\times\lim\limits_{h\to 0}\dfrac{\sin h}{h}$

$=\dfrac{k}{2}$ ....(i)

And $f\left(\dfrac{\pi}{2}\right)=3$ ....(ii)

$\because\ f(x)=3$ when $x=\dfrac{\pi}{2}$ [Given]

Because $f(x)$ is continuous at $x=\frac{\pi}{2}$

$\therefore \lim_{x\to\frac{\pi}{2}} f(x)=f\left(\frac{\pi}{2}\right)$

$\therefore$ From eq. (i) and (ii),

$\frac{k}{2}=3$

$\Rightarrow k=6$

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27. $f(x)=\begin{cases}kx^2, & \text{if } x\le 2\\ 3, & \text{if } x>2\end{cases}$ **at** $x=2.$

Ans. Here, $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-} kx^2= k\times 2^2$

$\because \lim_{x\to 2^+} f(x)=3$ and $f(2)=3$

Since $f(x)$ is continuous at $x=2$ [given]

Therefore, $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$

$\therefore k\times 2^2=3$

$\Rightarrow k=\frac{3}{4}$

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28. $f(x)=\begin{cases}kx+1, & \text{if } x\le \pi\\ \cos x, & \text{if } x>\pi\end{cases}$ **at** $x=\pi.$

Ans. Here, $\lim_{x\to\pi^-} f(x)=\lim_{h\to 0} f(\pi+h)=\lim_{h\to 0}\cos(\pi+h)=-\cos h=-\cos 0=-1$

And $\lim_{x\to\pi^-} f(x)=\lim_{h\to 0} f(\pi-x)=\lim_{h\to 0}\cos(\pi-h)=-\cos h=-\cos 0=-1$

Also $\lim_{x\to\pi} f(x) = \lim_{h\to 0}(k\pi+1)$

Since the given function is continuous at $x=\pi$

$\therefore \lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^-} f(x) = \lim_{x\to\pi} f(x)$

$\Rightarrow k\pi+1=-1$

$\Rightarrow k\pi=-2$

$\Rightarrow k=\frac{-2}{\pi}$

---

29. $f(x)=\begin{cases} kx+1, & \text{if } x\le 5 \\ 3x-5, & \text{if } x>5 \end{cases}$ **at** $x=5.$

Ans. When $x<5,$ we have $f(x)=kx+1$ which being a polynomial is continuous at each point $x<5.$

And, when $x>5,$ we have $f(x)=3x-5$ which being a polynomial is continuous at each point $x>5.$

Now $f(5)=5k+1=3(5+h)-5$

$\lim_{x\to 5^-} f(x) = \lim_{h\to 0} f(5+h) = 15+3h-5$ ....(i)

$= 10+3h = 10+3\times 0 = 10$

$\lim_{x\to 5^-} f(x) = \lim_{h\to 0} f(5-h) = k(5-h)+1 = 5k-hk+1 = 5k+1$ ....(i)

Since function is continuous, therefore, eq. (i) = eq. (ii)

$\Rightarrow 10=5k+1$

$\Rightarrow 5k=9$

$\Rightarrow k = \frac{9}{5}$

30. Find the values of a and b such that the function defined by $f(x) = \begin{cases} 5, & \text{if } x \le 2 \\ ax+b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \ge 10 \end{cases}$ is a continuous function.

Ans. For $x < 2$, function is $f(x) = 5$, constant, therefore it is continuous.

For $2 < x < 10$, function $f(x) = ax + b$, polynomial, therefore, it is continuous.

For $x > 10$, function is $f(x) = 21$, constant, therefore it is continuous.

For continuity at $x = 2$, $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$

$\Rightarrow \lim_{h \to 0} (5) = \lim_{h \to 0} \{a(2+h) + b\} = 5$

$\Rightarrow 2a + b = 5$ ....(i)

For continuity at $x = 10$, $\lim_{x \to 10^-} f(x) = \lim_{x \to 10^+} f(x) = f(10)$

$\Rightarrow \lim_{h \to 0} (21) = \lim_{h \to 0} \{a(10-h) + b\} = 21$

$\Rightarrow 10a + b = 21$ ....(ii)

Solving eq. (i) and eq. (ii), we get

$a = 2$ and $b = 1$.

31. Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

Ans. Let $f(x) = x^2$ and $g(x) = \cos x$, then

$(gof)(x) = g[f(x)] = g(x^2) = \cos x^2$

Now f and g being continuous it follows that their composite (gof) is continuous.

Hence $\cos x^2$ is continuous function.

32. Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

Ans. Given: $f(x) = |\cos x|$ ....(i)

$f(x)$ has a real and finite value for all $x \in$ R.

$\therefore$ Domain of $f(x)$ is R.

Let $g(x) = \cos x$ and $h(x) = |x|$

Since $g(x)$ and $h(x)$ being cosine function and modulus function are continuous for all real x

Now, $(goh)x = g\{h(x)\} = g(|x|) = \cos|x|$ being the composite function of two continuous functions is continuous, but not equal to $f(x)$

Again, $(hog)x = h\{g(x)\} = h(\cos x) = |\cos x| = f(x)$ [Using eq. (i)]

Therefore, $f(x) = |\cos x| = (hog)x$ being the composite function of two continuous functions is continuous.

33. Examine that $\sin|x|$ is a continuous function.

Ans. Let $f(x) = |x|$ and $g(x) = \sin|x|$, then

$(gof)x = g\{f(x)\} = g(|x|) = \sin|x|$

Now, f and g being continuous, it follows that their composite, (gof) is continuous.

Therefore, $\sin|x|$ is continuous.

34. Find all points of discontinuity of f defined by $f(x)=|x|-|x+1|$.

Ans. Given: $f(x)=|x|-|x+1|$

When $x<-1$, $f(x)=-x-\{-(x+1)\}$ = $-x+x+1=1$

When $-1\leq x<0$ $f(x)=-x-(x+1)=-2x-1$

When $x\geq 0$, $f(x)=x-(x+1)=1$

$$\therefore\ f(x)=\begin{cases} 1, & \text{if} \quad x<-1 \\ -2x-1, & \text{if} \quad -1\leq x<0 \\ -1, & \text{if} \quad x\geq 0 \end{cases}$$

At $x=-1$, L.H.L. = $\lim\limits_{x\to -1^-} f(x)=\lim\limits_{x\to -1^-} 1=1$

R.H.L. = $\lim\limits_{x\to -1^+} f(x)=\lim\limits_{x\to -1^+} (-2x-1)=1$

And $f(-1) = -2\times(-1)-1 = 1$

Therefore, at $x=-1$, $f(x)$ is continuous.

At $x=0$, L.H.L. = $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-} (-2x-1)=-1$

R.H.L. = $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+} (-1)=-1$

And $f(0)=-1$

Therefore, at $x=0$, $f(x)$ is continuous.

Hence, there is no point of discontinuity.